Lecturer: *Dana Randall*
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Abstract

We will prove the van den Berg-Kesten inequality in the form of Fishburn and Shepp following Reimer's proof in 1994.

We continue with the proof of the BK inequality due to Reimer. First, let us review the setup. Let X be a finite set, say $X = [n]$, where $[n] = \{1, 2, \ldots, n\}$. We consider a finite probability space (Ω, μ), where $\Omega = \{0,1\}^X$, $\mu = \mu_1 \times \mu_1 \cdots \times \mu_1$ and $\mu_1(\{1\}) = p, \mu_1(\{0\}) = 1 - p$.

Definition 1. *The **disjoint occurrence** of $A, B \subset \Omega$ is defined as*

$$A \circ B = \{x \in \Omega : \exists I = I(x) \subset [n], [x]_I \subset A, [x]_{I^c} \subset B\},$$

where

$$[x]_I = \{x' \in \Omega : x'_i = x_i \quad \forall i \in I\},$$

*is a **cylinder** and $I^c = [n] \setminus I$ is the complement of I in $[n]$.*

Theorem 1. *(BK inequality) For all $A, B \subset \Omega$*

$$\mu(A \circ B) \leq \mu(A)\mu(B)$$

We have shown the BK inequality if A and B are increasing events, that is, using FKG inequality as well,

$$\mu(A \circ B) \leq \mu(A)\mu(B) \leq \mu(A \cap B).$$

1 Equivalent Forms of the Inequality

We will show a couple of reductions that make the proof of the inequality possible. In the last lecture we proved a reduction, due to Fiebig and van den Berg, which states that we need to prove the BK inequality only in the case of $\Omega = \{0,1\}^n$ and uniform measure μ.

Proposition 2. *The BK inequality holds if for all n it is true for the uniform measure on $\Omega = \{0,1\}^n$, i.e. if for all $n \in \mathbb{N}$ and for all $A, B \subset \{0,1\}^n$*

$$|A \circ B|\, 2^n \leq |A|\, |B|\, .$$

Fishburn and Shepp derived another way of expressing the BK inequality and it is in this form that we will prove it.

Let $X \subset \Omega$, and let $S : X \to \{0,1\}^{[n]} : x \mapsto S(x) \subset [n]$ be an arbitrary map from X into $\{0,1\}^{[n]}$. We then define

$$[X]_S = \bigcup_{x \in X} [x]_{S(x)} \text{ and } [X]_{S^c} = \bigcup_{x \in X} [x]_{S(x)^c},$$

where, as before, $S(x)^c = [n] \setminus S(x)$. Now we will fix the disjoint occurence $X \equiv A \circ B$ of A and B and let the "cylinder sets" constructed above vary. We will see that the the product of the measures of these never falls below the mesure of the disjoint occurence and this turns out to be an equivalent statement for the BK inequality. The formal statement and its proof follow.

Proposition 3. *(Fishburn-Shepp). Consider $(\Omega, \mathcal{F}, \mu)$ as before. The the following two statements are equivalent:*

i) For all $A, B \in \mathcal{F}$,

$$|A \circ B|\,|\Omega| \le |A|\,|B|\,.$$

ii) For all $X \subset \Omega$ and all $S : X \to \{0,1\}^{[n]}$,

$$|X|\,|\Omega| \le |[X]_S|\,|[X]_{S^c}|\,.$$

Proof. i) $\Longrightarrow$ ii): Let $X \subset \Omega$, and let $S : X \to \{0,1\}^{[n]}$. Consider

$$A = [X]_S = \bigcup_{x \in X} [x]_{S(x)} \quad \text{and} \quad B = [X]_{S^c} = \bigcup_{x \in X} [x]_{S(x)^c}.$$

Then

$$X \subset A \circ B\,.$$

Hence, by i),

$$|X|\,|\Omega| \le |A \circ B|\,|\Omega| \le |A|\;|B| = |[X]_S|\;|[X]_{S^c}|\,,$$

which shows that i) implies ii).

ii) $\Longrightarrow$ i): Let $A, B \subset \Omega$. Let $X = A \circ B$. By the definition of $A \circ B$, for each $x \in X$ there is an $S(x)$ such that $[x]_{S(x)} \subset A$ and $[x]_{S(x)^c} \subset B$. We therefore have

$$[X]_S = \bigcup_{x \in X} [x]_{S(x)} \subset A$$

and

$$[X]_{S^c} = \bigcup_{x \in X} [x]_{S(x)^c} \subset B.$$

So, by ii),

$$|A \circ B|\,|\Omega| = |X|\,|\Omega| \le |[X]_S|\;|[X]_{S^c}| \le |A|\;|B|\,,$$

which shows that ii) implies i). □

2 Reimer's Main Lemma

We are in the special case when $\Omega = \{0,1\}^n$ and μ is the uniform measure on Ω. For $x \in \Omega$, denote by $\bar{x}$ the bitwise complement of x in Ω, that is $\bar{x}_i = 1 - x_i$, for all $i = 1, 2, \dots, n$. For a subset T of Ω, let $\bar{T} = \bigcup_{x \in T} \bar{x}$.

Lemma 4. *(Reimer's Main Lemma) Let $n \in \mathbb{N}$, $X \subset \Omega = \{0,1\}^n$ and $S : X \to \{0,1\}^{[n]} : x \mapsto S(x)$. Let*

$$U = [X]_S \qquad \text{and} \qquad V = [X]_{S^c}\,.$$

Then

$$\left|U \cap \bar{V}\right| \;=\; \left|\bar{U} \cap V\right| \;\ge\; |X|.$$

Now, if we prove that Reimer's main lemma implies the Fishburn-Shepp form of the BK inequality, we will be done because of Proposition 3.

Let us start by defining tensor product of two vectors. Let $\oplus$ denote concatenation given by, $(a,b)\oplus(c,d)=(a,b,c,d)$. Let $\otimes$ be the tensor product given by $(a,b)\otimes v = av\oplus bv$ for $a,b\in\mathbb{R}$ and $v\in\mathbb{R}^m$. Equipping $\mathbb{R}^{2^n}$ with the standard inner product: $\langle v\,|\,w\rangle=\sum_{i=1}^{2^n} v_iw_i$, notice that an easy inductive proof yields

$$\langle \bigotimes_{i=1}^n v^{(i)} \,|\, \bigotimes_{i=1}^n w^{(i)}\rangle = \prod_{i=1}^n \langle v^{(i)}\,|\,w^{(i)}\rangle \tag{1}$$

for $v^{(i)},w^{(i)}\in\mathbb{R}^2, 1\le i\le n$.

Indeed, we have that

$$\begin{aligned}
\langle \bigotimes_{i=1}^n v^{(i)} \,|\, \bigotimes_{i=1}^n w^{(i)}\rangle &= \langle (v_1^{(1)},v_2^{(1)})\otimes \bigotimes_{i=2}^n v^{(i)} \,|\, (w_1^{(1)},w_2^{(1)}) \bigotimes_{i=2}^n w^{(i)}\rangle \\
&= \langle (v_1^{(1)} \bigotimes_{i=2}^n v^{(i)}, v_2^{(1)} \bigotimes_{i=2}^n v^{(i)}) \,|\, (w_1^{(1)} \bigotimes_{i=2}^n w^{(i)}, w_2^{(1)} \bigotimes_{i=2}^n w^{(i)})\rangle \\
&= v_1^{(1)}w_1^{(1)} \langle \bigotimes_{i=2}^n v^{(i)} \,|\, \bigotimes_{i=2}^n w^{(i)}\rangle + v_2^{(1)}w_2^{(1)} \langle \bigotimes_{i=2}^n v^{(i)} \,|\, \bigotimes_{i=2}^n w^{(i)}\rangle \\
&= \langle v^{(1)}\,|\,w^{(1)}\rangle \langle \bigotimes_{i=2}^n v^{(i)} \,|\, \bigotimes_{i=2}^n w^{(i)}\rangle = \prod_{i=1}^n \langle v^{(i)}\,|\,w^{(i)}\rangle,
\end{aligned}$$

where the last equality is due to the induction hypothesis.

Proof. (Lemma 4 implies Theorem 1)

For $x,y\in\Omega$, let $\langle x,y\rangle$ be the cylinder $\langle x,y\rangle=\{z\in\Omega : z_i=x_i \text{ whenever } x_i=y_i\}$. Then

$$\begin{aligned}
|U|\,|V| &= |\{(u,v)\in U\times V\}| \\
&= \sum_A |\{(u,v)\in U\times V : \langle u,v\rangle = A\}| \\
&= \sum_A |\{(u,v)\in (U\cap A)\times (V\cap A) : \langle u,v\rangle = A\}|, \qquad (2)
\end{aligned}$$

where the sum runs over all cylinder sets $A\subset\Omega$. Defining

$$U_A = U\cap A \qquad \text{and} \qquad V_A = V\cap A$$

and observing that that $\langle u,v\rangle = A$ if and only if $u\in A$ and $v=\bar{u}^{(A)}$, which is the complement of u in A we get that

$$\begin{aligned}
|U|\,|V| &= \sum_A |\{(u,v)\in U_A\times V_A : v=\bar{u}^{(A)}\}| \\
&= \sum_A |U_A\cap \bar{V}_A^{(A)}|, \qquad (3)
\end{aligned}$$

where $\bar{V}_A^{(A)}$ is the complement of V_A in A.

We claim that Reimer's Main Lemma can be used to show that for each cylinder set $A \subset \Omega$

(4) $$|U_A \cap \bar{V}_A^{(A)}| \geq |X_A|, \qquad \text{where} \qquad X_A = X \cap A.$$

Intuitively this is clear - just apply the main lemma on a subcube. We skip the formal proof, it can be found in the expository paper.

Combining (3) and (4), we get

(5) $$|U|\,|V| \geq \sum_A |X \cap A|.$$

An easy counting argument gives that the right hand side of (5) is equal to $|X||\Omega|$. Indeed,

$$\sum_A |X \cap A| = \sum_A \sum_{x \in X \cap A} 1 = \sum_{x \in X} \sum_{A \ni x} 1 = |X||\Omega|,$$

which, together with (5), implies that

$$|U|\,|V| \geq |X||\Omega|,$$

the Fishburn-Shepp inequality for the uniform measure on $\Omega = \{0,1\}^n$. □

3 Proof of Reimer's Main Lemma

This proof follows an expository article by Borgs, Chayes and Randall, which in turn is based on the original proof by Reimer.

As a preparation, prove that a set of linearly independent vectors in $\mathbb{Z}_2^{2^n}$ is linearly independent in $\mathbb{R}^{2^n}$.

The first half of the statement of the main lemma just follows from the simple observation that $x \in U \cap \bar{V} \iff \bar{x} \in \bar{U} \cap V$. We therefore have to show that $|U \cap \bar{V}| \geq |X|$. Using de Morgan's laws, this is equivalent to showing that

$$|U^c \cup \bar{V}^c| \leq |\Omega| - |X|,$$

or

$$|U^c| + |U \cap \bar{V}^c| + |X| \leq |\Omega| = 2^n.$$

Since $|\bar{U}^c| = |U^c|$, this is equivalent to

$$|\bar{U}^c| + |U \cap \bar{V}^c| + |X| \leq |\Omega| = 2^n.$$

In order ot prove the last statement, we will construct injective maps α, β and γ from $\bar{U}^c$, $U \cap \bar{V}^c$ and X into $\mathbb{R}^{2^n}$.

We will show that the images of these maps are disjoint and that the union of the images is a set of linearly independent vectors in $\mathbb{R}^{2^n}$. This immediately implies that the number of elements in the union, and hence on the left hand side of the last ineqaulity, is bounded above by 2^n.

We begin by defining the maps α, β and γ in terms of a (still to be defined) function Φ as

$$\begin{aligned} \alpha &: \bar{U}^c \to \mathbb{R}^{2^n}: & x &\mapsto \Phi(x, \emptyset) \\ \beta &: U \cap \bar{V}^c \to \mathbb{R}^{2^n}: & x &\mapsto \Phi(x, [n]) \\ \gamma &: X \to \mathbb{R}^{2^n}: & x &\mapsto \Phi(x, S(x)). \end{aligned}$$

To define $\Phi(\cdot, S)$, we first define functions $\varphi_i(\cdot, S)$ on a single bit x_i:

$$\varphi_i(x_i, S) = \begin{cases} (x_i, -1) & \text{if} i \notin S \\ (1, x_i) & \text{if} i \in S. \end{cases} \tag{6}$$

With the tensor product notation in hand, let

$$\Phi(x, S) = \bigotimes_{i=1}^{n} \varphi_i(x_i, S)$$

for each $x \in \Omega$.

It suffices to verify the following six statements to show linear independence:

1. $\Phi(y, \emptyset) \perp \Phi(z, [n])$ for all $y \in \bar{U}^c$ and all $z \in U \cap \bar{V}^c$.
2. $\Phi(y, \emptyset) \perp \Phi(x, S(x))$ for all $y \in \bar{U}^c$ and all $x \in X$.
3. $\Phi(z, [n]) \perp \Phi(x, S(x))$, for all $z \in U \cap \bar{V}^c$ and all $x \in X$.
4. $\{\Phi(x, S(x)) : x \in X\}$ is linearly independent.
5. $\Phi(\bar{U}^c, \emptyset)$ is linearly independent.
6. $\Phi(U \cap \bar{V}^c, [n])$ is linearly independent.

The function Φ has been defined so that most of this will be routine.

(1) $\Phi(y, \emptyset) \perp \Phi(z, [n])$ for all $y \in \bar{U}^c$ and all $z \in U \cap \bar{V}^c$.
If $y \in \bar{U}^c$ and $z \in U \cap \bar{V}^c$, then $\bar{y} \notin U$ and $z \in U$, so in particular $\bar{y} \neq z$. Then $y_i = z_i$ for some i and

$$\langle \varphi_i(y_i, \emptyset) \,|\, \varphi_i(z_i, [n]) \rangle = \langle (y_i, -1) \,|\, (1, z_i) \rangle = 0.$$

We have that

$$\langle \Phi(y, \emptyset) \,|\, \Phi(z, [n]) \rangle = 0.$$

Since it is easy to see that neither $\Phi(y, \emptyset)$ nor $\Phi(z, [n])$ can be the zero vector, it follows that $\Phi(y, \emptyset) \perp \Phi(z, [n])$.

(2) $\Phi(y, \emptyset) \perp \Phi(x, S(x))$ for all $y \in \bar{U}^c$ and all $x \in X$.
If $y \in \bar{U}^c$ and $x \in X$, then $\bar{y} \notin U$ which implies there exists $i \in S(x)$ such that $y_i = x_i$. Thus, it follows that

$$\langle \varphi_i(y_i, \emptyset) \,|\, \varphi_i(x_i, S(x)) \rangle = \langle (y_i, -1) \,|\, (1, x_i) \rangle = 0.$$

Hence, $\Phi(y, \emptyset) \perp \Phi(x, S(x))$.

(3) $\Phi(z, [n]) \perp \Phi(x, S(x))$, for all $z \in U \cap \bar{V}^c$ and all $x \in X$.
If $z \in U \cap \bar{V}^c$ and $x \in X$, then $\bar{z} \notin V$ which implies there exists $i \in S(x)^c$ such that $z_i = x_i$. It follows that

$$\langle \varphi_i(z_i, [n]) \,|\, \varphi_i(x_i, S(x)) \rangle = \langle (1, z_i) \,|\, (x_i, -1) \rangle = 0.$$

Hence $\Phi(z, [n]) \perp \Phi(x, S(x))$.

(4) $\{\Phi(x, S(x)) : x \in X\}$ is a set of linearly independent vectors.

This statement is the core of Reimer's proof. For this argument, it is sufficient to prove the independence on $\mathbb{Z}_2^{2^n}$ rather than $\mathbb{R}^{2^n}$, and, as will become clear, it turns out to be much simpler for $\mathbb{Z}_2^{2^n}$. For the moment, simply note that, in $\mathbb{Z}_2^2$, if $x_i = 1$ then $\varphi_i(x_i, S) = (1,1)$ whether or not $i \in S$. Notice that since $X \subseteq \Omega$, we can think of $S : x \mapsto S(x)$ as a function from $X \to 2^{[n]}$. We can extend this by defining $S(x) \in \Omega$ for all $x \in \Omega \setminus X$ arbitrarily. This in turn induces a function $x \mapsto \Phi(x, S(x)) : \Omega \to \mathbb{R}^{2^n}$ (or $\mathbb{Z}_{\not\vDash}^{\not\vDash\ltimes}$) which coincides with γ when $x \in X$. In order to prove (4), it is therefore enough to prove that for all $S : \Omega \to 2^{[n]}$, the set $\{\Phi(x, S(x)) : x \in \Omega\}$ is a set of linearly independent vectors in $\mathbb{Z}_2^{2^n}$. This is the content of Lemma 5 below.

(5) $\Phi(\bar{U}^c, \emptyset)$ is linearly independent, and
(6) $\Phi(U \cap \bar{V}^c, [n])$ is linearly independent.

As an exercise, prove the last two statements independently.

Both of these statements follow as a special case of the statement that for all $S : \Omega \to 2^{[n]} : x \mapsto S(x)$, the set $\{\Phi(x, S(x)) : x \in \Omega\}$ is a set of linearly independent vectors in $\mathbb{R}^{2^n}$ (choose the constant functions $S(x) \equiv \emptyset$ and $S(x) \equiv [n]$, respectively).

The proof of Reimer's Main Lemma is therefore reduced to the proof of the following:

Lemma 5. *Let $n \in \mathbb{N}$, and let $\Phi : \{0,1\}^n \times 2^{[n]} \to \mathbb{R}^{2^n}$ be defined by (6) and (1). Let $S : x \mapsto S(x) \subset [n]$ be an arbitrary function from $\{0,1\}^n$ into $2^{[n]}$. Then the vectors $\Phi(x, S(x))$, $x \in \{0,1\}^n$, are linearly independent in $\mathbb{Z}_2^{2^n}$, and hence in $\mathbb{R}^{2^n}$.*

Proof. For $0 < k \leq 2^n$, let y^k be the configuration in Ω given by the binary representation of $k-1$ so that $\Omega = \{y^k : 0 < k \leq 2^n\}$, with $k = 1$ corresponding to $y_i^k \equiv 0$, $k = 2$ corresponding to $y_n^k = 1$ and $y_i^k = 0$ for all $i \leq n-1$, etc. For the configuration y^k in $\{0,1\}^n$, we let $0y^k$ be the configuration corresponding to the binary representation of $k-1$ in $\{0,1\}^{n+1}$, and $1y^k$ be the configuration corresponding to the binary representation of $2^n + k - 1$ in $\{0,1\}^{n+1}$.

If we let $A_S^{(n)}$ be the $2^n \times 2^n$ matrix formed by letting row k be the vector $\Phi(y^k, S(y^k))$,

$$A_S^{(n)}(k, \cdot) = \Phi(y^k, S(y^k))$$

then it suffices to show that for all functions $S : \Omega \to 2^{[n]}$, the matrix $A_S^{(n)}$ satisfies

$$\det A_S^{(n)} = 1.$$

We will prove this using induction on n. The base case $n = 1$ is trivial to check. So suppose that for all $S : \{0,1\} \to 2^{[n]}$ we have $\det A_S^{(n)} = 1$ by induction. Analyzing the case $n+1$, let now $\Omega = \{0,1\}^{n+1}$, and let S be a function from $\{0,1\}^{n+1}$ into $2^{[n+1]}$. Note that the binary representation of each of the first 2^n configurations begins with 0. So $\varphi_1(y_1^k, S(y^k)) = (1,0)$ or $(0,-1)$ (which equals $(0,1)$ in $\mathbb{Z}_2$), depending on whether $1 \in S(y^k)$ or not. Therefore, defining $S^0 : \{0,1\}^n \to 2^{[n]}$ by $S^0(y^k) = \{i \in [n] : i+1 \in S(0y^k)\}$, we get that for each $0 \leq k < 2^n$, either $1 \in S(y^k)$ and

$$\begin{aligned} A_S^{(n+1)}(k, \cdot) &= (1,0) \otimes \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)) \\ &= \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)) \oplus \bigoplus_{j=1}^{2^n} 0 \\ &= A_{S^0}^{(n)}(k, \cdot) \oplus \bigoplus_{j=1}^{2^n} 0 \end{aligned}$$

or $1 \notin S(y^k)$ and

$$\begin{aligned} A_S^{(n+1)}(k,\cdot) &= (0,1) \otimes \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)) \\ &= \bigoplus_{j=1}^{2^n} 0 \oplus \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)). \\ &= \bigoplus_{j=1}^{2^n} 0 \oplus A_{S^0}^{(n)}(k,\cdot) \ . \end{aligned}$$

Defining $\varepsilon_k = 1$ if $1 \in S(y^k)$ and $\varepsilon_k = 0$ if $1 \notin S(y^k)$, we therefore have that

$$A_S^{(n+1)}(k,\cdot) = \varepsilon_k A_{S^0}^{(n)}(k,\cdot) \oplus (1-\varepsilon_k) A_{S^0}^{(n)}(k,\cdot).$$

Meanwhile, note that $(1,-1) = (1,1)$ in $\mathbb{Z}_2^2$, so that $\varphi_1(y_1^k, S(y^k)) = (1,1)$ if the binary representation of k starts with 1. Therefore, defining $S^1 : \{0,1\}^n \to 2^{[n]}$ by $S^1(y^k) = \{i \in [n] : i+1 \in S(1y^k)\}$, we get that for each $2^n < k \leq 2^{n+1}$

$$\begin{aligned} A_S^{(n+1)}(k,\cdot) &= (1,1) \otimes \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)) \\ &= \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)) \oplus \bigotimes_{i=2}^{n+1} \varphi_i(y_i^k, S(y^k)) \\ &= A_{S^1}^{(n)}(k,\cdot) \oplus A_{S^1}^{(n)}(k,\cdot) \ . \end{aligned}$$

Hence

$$A_S^{(n+1)} = \begin{pmatrix} \varepsilon_k A_{S^0}^{(n)}(k,\cdot) & (1-\varepsilon_k) A_{S^0}^{(n)}(k,\cdot) \\ \\ A_{S^1}^{(n)} & A_{S^1}^{(n)} \end{pmatrix}$$

Although this matrix looks messy, a few column operations—actually 2^n of them—will improve things, without changing the determinant, of course. By adding column $k+1$ to column $k+1+2^n$ (for each $0 \leq k < 2^n$) which, in $\mathbb{Z}_2$, is the same as subtracting column $k+1$ from column $k+1+2^n$, we can conclude that

$$\begin{aligned} \det A_S^{(n+1)} &= \det \begin{pmatrix} \varepsilon_k A_{S^0}^{(n)}(k,\cdot) & A_{S^0}^{(n)}(k,\cdot) \\ \\ A_{S^1}^{(n)} & 0 \end{pmatrix} \\ &= \det A_{S^0}^{(n)} \det A_{S^1}^{(n)} \\ &= 1, \end{aligned}$$

where the final step follows by induction. □

This completes the proof of Reimer's Main Lemma, and hence the proof of the BK inequality, Theorem 1.