**Math 8223A – Combinatorial Methods for Statistical Physics Models**

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**Abstract**

In these lectures, we present the Random Cluster model and the Swendsen-Wang sampling algorithm.

# 1 The Random Cluster model

In the previous lecture, we discussed the setup of the Ising model and presented Peierl's argument to show the existence of a phase transition at a critical temperature, and at least two Gibb's states below that temperature.

First, let us make a change of notation, so that our model generalizes more easily to for instance the *Pott's model*, in which, instead of two possible states, + and −, there might be $q$ states $\{1, 2, \ldots, q\}$.

Again, for convenience, assume $h \equiv 0$, since we can always consider an extra vertex which is adjacent to every other vertex in the lattice with interaction strength being the external magnetic field $h$ Now we make a change of variables defined thus:

$$\begin{aligned} J'_{ij} &= \frac{J_{ij}}{2} \\ Z' &= Z \prod_{i \sim j} e^{-\beta J'_{ij}} \\ H'(\sigma) &= -\sum J'_{ij}(\delta_{\sigma_i \sigma_j} - 1) \\ \pi'(\sigma) &= \frac{e^{-\beta H'(\sigma)}}{Z'} \end{aligned}$$

(For ease of notation, we will now drop the primes).

**Claim:** For a given edge $e = (x, y)$,

$$e^{\beta J_{xy}(\delta_{\sigma_x \sigma_y} - 1)} = 1 - P_{xy} + P_{xy}\delta_{\sigma_x \sigma_y} \text{ where } P_{xy} = e^{-\beta J_{xy}}$$

This can be seen by considering the two cases, when $\sigma_x = \sigma_y$ and otherwise.

Now,

$$\begin{aligned} Z = \sum_\sigma \pi(\sigma) &= \frac{1}{Z} \prod_{x \sim y} e^{\beta J_{xy}(\delta_{\sigma_x \sigma_y} - 1} \\ &= \prod_{x \sim y} \underbrace{(1 - P_{xy})}_{(x,y) \notin W} + \underbrace{P_{xy}\delta_{\sigma_x \sigma_y}}_{(x,y) \in W} \text{ , where W} \subseteq \text{set of bonds} \\ &= \sum_{W \subseteq edges E} \prod_{(x,y) \notin W} (1 - P_{xy}) \prod_{(x,y) \in W} P_{xy}\delta_{\sigma_x \sigma_y} \\ &= \sum_{W \subseteq edges} \prod_{(x,y) \notin W} (1 - P_{xy}) \prod_{(x,y) \in W} P_{xy} \prod_{(x,y) \in W} \delta_{\sigma_x \sigma_y} \end{aligned}$$

Since $\delta_{\sigma_x \sigma_y}$ varies over all edges in $\Lambda$ and is 1 if and only if the end vertices have the same sign, using this, we are counting all the edges whose endpoints have the same sign.

$$\begin{aligned} Z &= \sum_{\sigma}\left(\sum_{\omega\in E}\prod_{(x,y)\notin\omega}(1-P_{xy})\prod_{(x,y)\in\omega}P_{xy}\prod_{(x,y)\in\omega}\delta_{\sigma_x\sigma_y}\right) \\ &= \sum_{\sigma}\sum_{\omega\in E}\prod_{(x,y)\notin\omega}(1-P_{xy})\prod_{(x,y)\in\omega}P_{xy}2^{\sharp(\omega)} \end{aligned}$$

We use $2^{\sharp(\omega)}$ since we are representing the Ising model, for Pott's model, we would have $q^{\sharp(\omega)}$ where $\sharp(\omega)$ is the number of components in $\omega$.
This is called the **Random cluster model** or the **Fortuin-Kastelyn representation**.

## 2 Swendsen-Wang Algorithm

We will first describe another representation for $Z$ which will be useful when we talk about the Swendsen-Wang algorithm. In the following expression,$\omega$ varies over the subsets of the edges of the lattice (as above) and $\omega_b$ represents the value of an edge $b$ So $\omega_b$ takes the value 0 if it connects vertices $x$ and $y$ such that$\sigma_x \neq \sigma_y$ and 1 otherwise, and $P_b$ is the same as $P_{xy}$ where $b$ is the edge connecting $x$ and $y$.

$$\begin{aligned} Z &= \sum_{\sigma}\sum_{\omega}\left[\prod_{b:\omega_b=0}(1-P_b)\prod_{b:\omega_b\neq 0}P_b\prod_{(x,y)\in\omega}\delta_{\sigma_x\sigma_y}\right] \\ &= \sum_{\sigma}\sum_{\omega}\prod_{b\in edges}\left[((1-P_b)(1-\delta_{b\in\omega})+P_b\delta_{b\in\omega})\delta_{\sigma_x\sigma_y}\right], \\ f(\omega,\sigma) &= \prod_{b:\omega_b=0}(1-P_b)\prod_{b:\omega_b\neq 0}P_b\prod_{(x,y)\in\omega}\delta_{\sigma_x\sigma_y} \\ &= ((1-P_b)(1-\delta_{b\in\omega})+P_b\delta_{b\in\omega})\delta_{\sigma_x\sigma_y}. \end{aligned}$$

The Swendsen-Wang process is used to sample from configurations in order to reach the equilibrium or Gibbs distribution. To use this algorithm, we first list all the configurations $\sigma$ and all the subgraphs $\omega$ of the finite lattice. For each $\sigma$, we let $B(\sigma)$ be the set of bonds between vertices with like signs. This would define a subgraph of the lattice. We can get the "marginals" of the joint distribution of $\omega$ and $\sigma$ by summing $f(\omega,\sigma)$ over $\sigma$ and $\omega$ respectively and dividing by $Z$. These are the steps we follow:
• Start with some configuration $\sigma$
• For each edge in $B(\sigma)$, remove the edge with probability $1 - P_{xy}$ and leave it in with probability $P_{xy}$.
• This gives us a random subgraph W.
• Choose $\sigma'$ consistent with $W$, by flipping a fair coin to assign signs to each component.
• Repeat