Abstract

In these lectures, we will discuss the Ising model, define Gibbs states and describe Peierl's argument to show that there is a phase transition in the two-dimensional Ising Model.

1 The Ising Model : Definitions and set-up

The Ising model is a model for ferromagnetism and in two dimensions demonstrates physics of phase transition. (A phase transition occurs when a small change in a parameter such as temperature causes a large scale *qualitative* change in the state of the system).

We consider a lattice region Λ (assume Λ is finite) and a configuration $\sigma \subseteq \{+,-\}^{\Lambda}$ (That is, σ is an assignment of values $\sigma = (\sigma_1, \sigma_2, ...)$ to each vertex in the lattice, so that each vertex is assigned a $+$ or a $-$. We could also think of this as each vertex being assigned a positive or negative spin, which can be denoted as a $+1$ or a -1). Next we define the Hamiltonian of the system. Here, we assume that only "nearest neighbors" interact, so the only interactions that contribute to the Hamiltonian are the interactions between adjacent vertices and the interaction of all the vertices with an external field.

Given a configuration σ,the **Hamiltonian** is defined as:

$$H(\sigma) = -\sum_{i \sim j} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i$$

The first sum is over all pairs of adjacent vertices and the second sum is over all vertices. J_{ij} is the energy or interaction strength associated with the the interactions between adjacent vertices and h represents the interactions with the external field. For the ferromagnetic model, $J_{ij} > 0$ which implies that the configuration with most $\sigma_i = \sigma_j$ is favored. (For an anti-ferromagnetic model $J_{ij} < 0$).

We now define the **Partition function** Z which will be essential in defining a probability distribution on the configurations.

$$Z = \sum_{\sigma} e^{-\beta H(\sigma)}$$

where we sum over all possible configurations σ; and $\beta = \frac{1}{kT}$ where k is Boltzmann's constant and T is absolute temperature.

We next define a probability distribution on the set of all configurations, so that the probability of a configuration σ is :

$$\pi(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}$$

The negative sign in the exponent implies that configurations with lower $H(\sigma)$ (or lower energy), are more likely. In other words, if β is small, which happens at high temperatures, then spins are almost independent, and if β is large, (at low temperatures), then clusters of $+$'s and $-$'s are more likely.

Intuitively,this means that at zero external field, as temperature decreases, we see configurations with approximately equal numbers of $+$'s and $-$'s until a critical temperature t_c at which we see a phase transition with dominantly $+$'s or $-$'s. We are really interested in defining a probability measure on configurations of the infinite lattice and will do this by defining appropriate limits. The *thermodynamic limit* is defined as a measure π defined by a conditional probability on finite lattice regions Λ. (This is the "so called" Gibbs measure).

It will be useful to study the infinite lattice $\mathbb{Z}^2$ and consider configurations on some finite part Λ, given a fixed configuration outside which we will call η.

Let $S = \mathbb{Z}^2$ and define the conditional probability of a configuration on Λ as :

$$\begin{aligned} \pi(\sigma\, on\, \Lambda \,|\, \eta\, on\, S\setminus\Lambda) &= \frac{\pi(\sigma\eta\, on\, S)}{\pi(\eta\, on\, S\setminus\Lambda)} \\ &= \frac{e^{-\beta H(\sigma\eta)}}{Z(\Lambda,\eta)} \qquad (1) \end{aligned}$$

where $Z(\Lambda, \eta)$ is the normalizing constant that depends on Λ and the outside configuration η.

Definition 1. *π is a **Gibbs state** if it satisfies* (1) *for all η, Λ.*

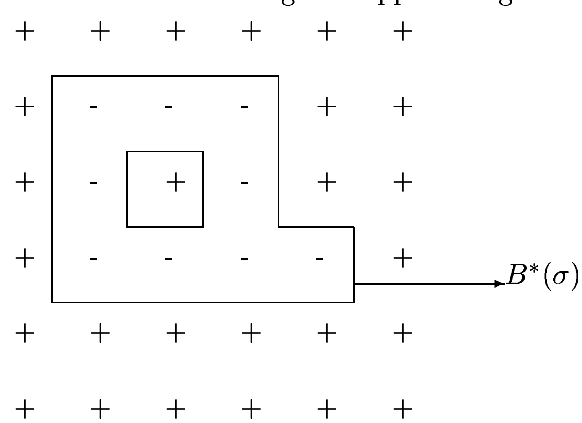
Thus Gibbs states are probability distributions and now an important question comes up: Is π unique? We will see that it is *not* unique below the critical temperature.

Note that when we compute π it only matters what we assign to the boundary of Λ. Since the other vertices in $S\setminus\Lambda$ are not adjacent to vertices in Λ, these interactions do not affect π. We will consider two cases for η :
(i) $\eta = (+)^{S\setminus\Lambda}$ and call this limiting distribution π^+
(ii)$\eta = (-)^{S\setminus\Lambda}$ and call this π^-

We will see that these limiting distributions are the same above the critical temperature and different below it.

Theorem 1. *(Peierl's) $\exists$ a β such that $\pi^+ \neq \pi^-$. That is, there are at least two Gibbs states.*

Assume that $h = 0$. For a configuration σ, Let B be the set of bonds in Λ and B^* the set of dual bonds. Let $B^*(\sigma)$ be the dual bonds that cross edges of opposite sign.



Notice that if the boundary $\eta = +^{S\setminus\Lambda}$ and the origin is $-$, then there must be a circuit in $B^*(\sigma)$ separating the origin from the boundary. Now, we can rewrite the Hamiltonian as :

$$\begin{aligned} H(\sigma) &= -\sum_{i\sim j} \sigma_i\sigma_j \;\; \text{where}\, \sigma_i\sigma_j = \begin{cases} 1 & \text{if}\, \sigma_i = \sigma_j \\ -1 & \text{otherwise} \end{cases} \\ &= -|B| + 2|B^*(\sigma)| \\ &= -|B^*| + 2|B^*(\sigma)| \end{aligned}$$

First, fix $\eta = +^{S\setminus\Lambda}$ (All + on the boundary). Fix σ s.t. $\sigma(0) = -1$ Let D^+ be the largest connected component of -1's containing the origin and let D^- be the unique infinite component in the complement $S\setminus D^+$.

Claim 1. *The set of dual bonds which crosses edges(i,j) s.t. $i \in D^-$ and $j \in D^+$ forms a circuit.*

Fix circuit C (containing the origin).
Let A be the set of configurations with all +'s outsideΛ.
Let $A_1 \subseteq A$ be the set of configurations $\hat\sigma$ such that $C \in B^*(\hat\sigma)$ and let $A_2 \subseteq A$ be the set of configurations $\hat\sigma$ such that $C \cap B^*(\hat\sigma) = \emptyset$. We will show that $|A_1| = |A_2|$.

Define a map :

$$\tau_C(\sigma_i) = \begin{cases} \sigma_i & \text{if } i \text{ is outside } C \\ -\sigma_i & \text{if } i \text{ is inside } C \end{cases}$$

Then

$$\tau_C(\sigma_i)\tau_C(\sigma_j) = \begin{cases} -\sigma_i\sigma_j & \text{if } (i,j)^* \in C \text{ where } (i,j)^* \text{ is the dual bond to } (i,j) \\ \sigma_i\sigma_j & \text{otherwise} \end{cases}$$

Thus, the only bonds which change are *on* the circuit.
Therefore, τ_C is a bijection between A_1 and A_2 and $|A_1| = |A_2|$.
In particular, the map τ gives us that:

$$\begin{aligned} H(\sigma) - H(\tau_C(\sigma)) &= -|B^*| + 2|B^*(\sigma)| - (-|B^*| + 2|B^*(\tau_C(\sigma))|) \\ &= 2(|B^*(\sigma)| - |B^*(\tau_C(\sigma))|) \\ &= 2|C|. \end{aligned} \tag{2}$$

With this observation, we can now examine the probability that any particular circuit C is in $B^*(\sigma)$. This will be the basis of Peierl's argument.

Lemma 2. $\pi(C \subseteq B^*(\sigma)\,|\,\eta = +) \;\; < e^{-2\beta|C|}$

Proof.

$$\begin{aligned} \pi(C \subseteq B^*(\sigma)) &= \frac{\sum_{\sigma\in A_1} e^{-\beta H(\sigma)}}{\sum_{\sigma\in A} e^{-\beta H(\sigma)}} \\ &\le \frac{\sum_{\sigma\in A_1} e^{-\beta H(\sigma)}}{\sum_{\sigma\in A_2} e^{-\beta H(\sigma)}} \\ &= \frac{\sum_{\sigma\in A_2} e^{-\beta H(\tau_C(\sigma))}}{\sum_{\sigma\in A_2} e^{-\beta H(\sigma)}} \\ &= \frac{\sum_{\sigma\in A_2} e^{-\beta H(\sigma)-2\beta|C|}}{\sum_{\sigma\in A_2} e^{-\beta H(\sigma)}} \\ &= e^{-2\beta|C|} \end{aligned}$$

□

Therefore, from (2) and Lemma(2), we have:

$$\begin{aligned}\pi(\sigma(0) = -1 \,|\, \eta = +) &\leq \sum_{circuits C} \pi(C \in B^*(\sigma) \,|\, \eta = +) \\ &\leq \sum_{C} e^{-2\beta|C|} \\ &\leq \sum_{l \geq 1} l3^l e^{-2\beta l}\end{aligned}$$

(Since $l3^l \geq$ the number of circuits around the origin).

This shows that the limit of $\pi(\sigma(0) = -1 \,|\, \eta = +)$ as $\beta \to \infty$ is zero. So there is some β for which $\pi(\sigma(0) = -1 \,|\, \eta = +) < \frac{1}{2}$ for any finite Λ.Thus, no matter how large Λ is, the probability that the origin is -1 is small if the boundary of Λ is $+$. (For instance, a simple calculation shows that if $\beta > 2\log 3$ then $\pi(\sigma(0) = -1 \,|\, \eta = +) < \frac{1}{200}$.) This was *Peierl's argument* to show that there exists a phase transition at some temperature t_c below which observations at the origin will distinguish π^+ from π^-.

Conclusion Thus, we have seen that if the boundary of the finite lattice has all +'s then $Pr(origin = +) \geq \frac{1}{2}$,no matter how large the lattice is, which shows that $\pi^+ \neq \pi^-$ and there exist at least two distinct Gibbs states for the 2-dimensionsal Ising model.