Math 8223A – Combinatorial Methods for Statistical Physics Models

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Today we introduce some basics of Percolation Theory and the FKG Inequality.

1 Percolation

Let $\Lambda = \mathcal{Z}^d$ denote the integer lattice and $\mathcal{B}$ denote the set of bonds (edges) in the lattice. Let $\Omega = \{0,1\}^{\mathcal{B}} = \prod_{b \in \mathcal{B}} \{0,1\}$. For $\omega \in \Omega$, $\omega_b = 1$ means that the bond b is open (occupied), and $\omega_b = 0$ means that bond b is closed (unoccupied). Also, for $\omega \in \Omega$, we let $S(\omega) = \{b : \omega_b = 1\}$ be the collection of bonds which are open. For x a vertex of Λ, we let $c(x)$ denote the set of vertices connected to x using edges of $S(\omega)$. (I.e. $c(x)$ is the connected component containing x.) We write $x \leftrightarrow y$ if $c(x) = c(y)$ (meaning that x and y are in the same connected component), and $x \leftrightarrow \infty$ if $|c(x)| = \infty$.

Definition 1 *Let $0 \le p \le 1$ and for any bond b let $\mu_b(1) = p$, $\mu_b(0) = q = 1 - p$. We define the percolation measure on a configuration, ω, to be*

$$P_p(\omega) = \prod_{b \in \mathcal{B}} \mu_b(\omega_b).$$

The primary questions that we are interested in are when is it likely that $x \leftrightarrow y$, and, more importantly, when is it likely that $0 \leftrightarrow \infty$?

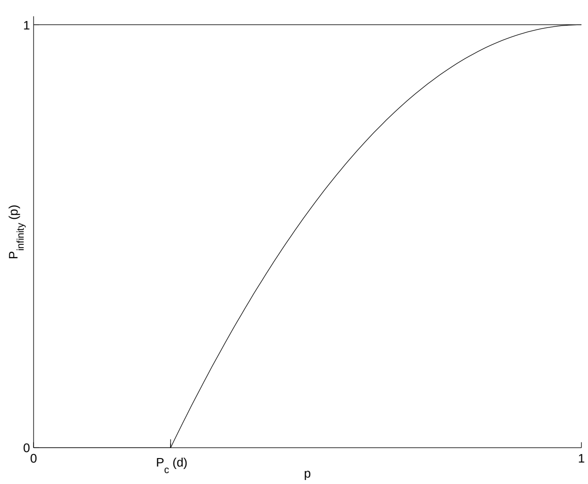
Definition 2 *Let $P_\infty(p) = P_p(0 \leftrightarrow \infty)$. If $P_\infty(p) > 0$ we say that we have percolation.*

Remarks:

1. $P_\infty(0) = 0$, so there is no percolation.
2. $P_\infty(1) = 1$, so there is percolation.
3. Intuitively, if $P_\infty(p) > 0$, then $P_\infty(p') > 0$ for $p' > p$.

Definition 3 *In $\mathcal{Z}^d$, let $P_c(d) = \inf\{p : P_\infty(p) > 0\}$.*

A sketch of $P_\infty(p)$ versus p is given below.



We will prove that this "critical" probability is strictly between zero and one. First, some more definitions that we will use to do this.

Let $\sigma_d(N)$ be the number of self avoiding walks of length N in $\mathcal{Z}^d$.
Let

$$\lambda(d) = \lim_{N\to\infty} \sigma_d(N)^{\frac{1}{N}}.$$

Note that we have $d \le \lambda(d) \le 2d-1$.

To see the lower bound, start at the origin, and construct a self avoiding walk by choosing one of the d coordinates and adding 1 to that coordinate (i.e. always walk in the positive direction). Do this for N steps. The walk constructed in this manner is self avoiding since each coordinate position is a non-decreasing function of N (and one coordinate increases at each step). The number of walks we can construct this way is d^N, hence $\sigma_d(N) \ge d^N$.

For the upper bound, start at the origin, choose one of the $2d$ vertices connected to the origin, and move there. For each step after that, choose one of the $2d-1$ other vertices joined to the current one, and move there (i.e. don't return to the one you just came from). The number of walks constructed in this manner is $2d(2d-1)^{N-1}$, so $\sigma_d(N) \le 2d(2d-1)^{N-1}$.

Theorem 1 *For $d \ge 2$ we have $0 < P_c(d) < 1$. More specifically, $\frac{1}{\lambda(d)} \le P_c(d) \le 1 - \frac{1}{\lambda(2)}$.*

Proof: ($P_c(d) \ge \frac{1}{\lambda(d)}$)
The probability that any self avoiding walk of length N consists of all open bonds is p^N. For any configuration let $\tau_d(N)$ be the number of self avoiding walks of length N which are open. Then $E_p(\tau_d(N)) = p^N\sigma_d(N)$. So

$$\begin{aligned} P_\infty(p) &\le P_p(\tau_d(N) \ge 1) \qquad \text{(for all } N \ge 1) \\ &\le E_p(\tau_d(N)) = p^N\sigma_d(N) \end{aligned}$$

Note that $p < \frac{1}{4}$ suffices for the limit to be 0. In fact, as $N \to \infty$, we have that $P_\infty(p) \to 0$. Therefore , $P_\infty(p) \ge \frac{1}{\lambda(d)}$.

($P_c(d) \le 1 - \frac{1}{\lambda(2)}$)
First observe that showing this statement in 2-d is sufficient as $P_c(d) \le P_c(2)$ for $d > 2$ (since if we have percolation on some 2-d sub-lattice containing the origin, then we have percolation in the d dimensional lattice as well). Therefore, consider the restriction to two dimensions.

For any configuration ω, if we don't have percolation, then there is a simply connected circuit in the dual lattice enclosing the component that contains the origin, where all of the bonds crossing the circuit are closed. Let Γ_N be the number of simply connected circuits of length N enclosing the origin. Then $\Gamma_N \le N \cdot \sigma_d(N-1)$. (Why? Any circuit of length N must cross the x-axis between $(0,0)$ and $(N,0)$. Then starting at $(x+\frac{1}{2}, \frac{1}{2})$ in the dual lattice, the next $N-1$ edges of the circuit must be self avoiding.) So

$$\sum_{\gamma:\text{circuit}} P_p(\gamma \text{ closed}) \le \sum_{N=1}^{\infty} q^N N\sigma_d(N-1).$$

(Here we note that if $\lambda(2)\cdot q < 1$ then the probability of any closed circuit is finite. In fact, as $q \to 0$, $P_p(\text{closed circuit}) \to 0$.)

Let F_N be the event of no closed cicuit of length $\leq N$, and let G_N be the event of no closed circuit of length $\geq N$.
Then

$$\begin{aligned} P_\infty(p) &\geq P_p(F_N \cap G_N) \\ &= P_p(F_N|G_N)P_p(G_N) \\ &\geq P_p(F_N)P_p(G_N) \end{aligned}$$

where the last inequality follows from the FKG inequality, which we prove below. If $q < \frac{1}{\lambda(2)}$, there's some value of N such that the probability of a closed circuit of length at least N is at most $\frac{1}{2}$, i.e. $P_p(G_N^C) < \frac{1}{2}$ for some N. But for any finite N we have $P_p(F_N) > 0$, hence $P_\infty(p) > 0$ (if $q < \frac{1}{\lambda(2)}$). $\square$

2 The FKG Inequality

We will prove the FKG inequality on finite state spaces, although it holds for general percolation spaces.

Consider the state space $\Omega = \mathcal{Z}_n^d$, and $A \subset \Omega$ is an "event."

Definition 4 *A is an increasing event if for all $\omega \in A$, if $\omega' > \omega$ then $\omega' \in A$.*

Examples

1. $A = \{\omega : 0 \leftrightarrow \infty\}$ on the infinite lattice $\mathcal{Z}^d$.
2. $A = \{\omega : \omega \text{ has a left-right crossing}\}$ in $\mathcal{Z}_n^2$.

The FKG inequality was discovered by Harris, and by Fortuin, Kasteleyn, and Ginibre.

Theorem 2 (FKG) *If A and B are increasing events, then*

$$P_p(A \cap B) \geq P_p(A)P_p(B).$$

We will prove a stronger version of this theorem. First we need a definition.

Definition 5 *Let μ be a probability mesaure on Ω. We say that μ satisfies the FKG condition if*

$$\mu(a \cup b)\mu(a \cap b) \geq \mu(a)\mu(b) \quad (1)$$

for all $a, b \in \Omega$.

We will prove a stronger version of the FKG inequality due to Holley.

Theorem 3 (FKG, Stronger Version) *Let μ be a probability measure on Ω that satisfies the FKG condition (1), and let f and g be increasing functions on Ω. Then*

$$\sum_{a\in\Omega} f(a)g(a)\mu(a) \geq \sum_{a\in\Omega} f(a)\mu(a) \sum_{b\in\Omega} g(b)\mu(b).$$

Remarks:

1. P_p is an FKG measure (i.e. it satisfies the FKG condition (1), with equality)

$$\frac{P_p(a \cup b)}{P_p(a)} = \left(\frac{p}{1-p}\right)^{|b \backslash a|} = \frac{P_p(b)}{P_p(a \cap b)}$$

   where $b \backslash a$ are the bonds which are open in b but not in a.

2. Let $f = 1_A$, i.e. the indicator for the set A, and $g = 1_B$. Then, with these f and g, the stronger version (Thm. 3) implies the first FKG inequality (Thm. 2).

In the next lecture we will see the proof of the FKG inequality.