1. Negate the following sentences:

a) For all integers \( n \geq 4 \), there exists \( c \in \mathbb{R} \) such that \( n^{100} \geq 2^n \).

Ans: There exists an integer \( n \), \( n \geq 4 \), such that for all \( c \in \mathbb{R} \), \( n^{100} < 2^n \).

b) The square of an integer is never odd.

Ans: There exists an integer \( n \) such that \( n^2 \) is odd.

c) \( \forall x \in \mathbb{R} \forall y \in \mathbb{R} \ [x > y] \) or \( [x < y] \).

Ans: \( \exists x \in \mathbb{R} \exists y \in \mathbb{R} \ [x \geq y] \) and \( [x \leq y] \).

d) \( \forall x \in \mathbb{Z}^+ \exists y \in \mathbb{Z}^+ \ [x^2 = y^2] \) and \( [x \neq y] \).

Ans: \( \exists x \in \mathbb{Z}^+ \forall y \in \mathbb{Z}^+ \ [x^2 \neq y^2] \) or \( [x = y] \).

2. **Theorem 1.** For all \( n \in \mathbb{Z}^+ \),
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Proof.** (by induction)

**Base case:** \( n = 1 \):
\[
\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6}.
\]

**Induction hypothesis:** Let \( k \geq 1 \) and assume
\[
\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.
\]

**Inductive step:** We will show the equation holds when \( n = k + 1 \).
\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the induction hypothesis)}
\]
\[
\begin{align*}
\frac{k+1}{6}(k(2k+1)+6(k+1)) &= \frac{k+1}{6}(2k^2+7k+6) = \frac{(k+1)(k+2)(2(k+1)+1)}{6}.
\end{align*}
\]

Thus, by induction, for all \(n \in \mathbb{Z}^+\),
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]

3. Let \(f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}\) be defined by
\[
f(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)
\]
for all reals \(x_1, x_2\).

Prove that \(f\) is invertible.

**Proof.** First we will show that \(f\) is onto, i.e., for all \((a, b) \in \mathbb{R} \times \mathbb{R}\), \(\exists (x, y) \in \mathbb{R} \times \mathbb{R}\) such that \(f(x, y) = (a, b)\). Let \((a, b) \in \mathbb{R} \times \mathbb{R}\). Let \(x = \frac{a+b}{3}\) and let \(y = \frac{2a-b}{3}\), where both \(x\) and \(y\) are reals since \(a\) and \(b\) are. Then
\[
f(x, y) = f\left(\frac{a+b}{3}, \frac{2a-b}{3}\right) = \left(\frac{a+b+2a-b}{3}, \frac{2a+2b-(2a-b)}{3}\right) = (a, b).
\]
Hence \(f\) is onto.

Now we will show that \(f\) is 1-1. Suppose for \(x, y, x', y' \in \mathbb{R}\), \(f(x, y) = f(x', y')\). Then \(x + y = x' + y'\) and \(2x - y = 2x' - y'\). Adding these two equations, we find \(x + y + 2x - y = x' + y' + 2x' - y'\) which implies \(3x = 3x'\) and hence \(x = x'\). Substituting this into the first equation, we find \(x + y = x + y'\) or \(y = y'\). Therefore \(f(x, y) = f(x', y')\) implies \((x, y) = (x', y')\) and so \(f\) is 1-1.

Since we have shown that \(f\) is both onto and 1-1, it is invertible.

4. Let \(a_1 = 5\), \(a_2 = 13\), and, for \(n \geq 2\), let \(a_{n+1} = 5a_n - 6a_{n-1}\).

Prove the following theorem.

**Theorem 2.** For all \(n \in \mathbb{Z}^+\), \(a_n = 3^n + 2^n\).

**Proof.** (by strong induction)

**Base cases:** \(n = 1: a_1 = 3^1 + 2^1 = 5\). \(n = 2: a_2 = 3^2 + 2^2 = 13\).

**Induction hypothesis:** Let \(k \geq 2\) be an integer and assume that \(\forall n \in \mathbb{Z}, 0 \leq n \leq k, \ a_n = 3^n + 2^n\).
Inductive step: We want to show that $a_{n+1} = 3^{n+1} + 2^{n+1}$.

\[ a_{n+1} = 5a_n - 6a_{n-1} = 5(3^n + 2^n) - 6(3^{n-1} + 2^{n-1}) \]
\[ = (15 - 6)3^{n-1} + (10 - 6)2^{n-1} = 3^{n+1} + 2^{n+1}. \]

Hence, by induction, $a_n = 3^n + 2^n$ for all integers $n \geq 1$.

5. Prove that for all integers $n \geq 1$, $8^n - 2^n$ is a multiple of 6.

Proof. (by induction) Base case: If $n = 1$, then $8^1 - 2^1 = 6$ so it is a multiple of 6. Inductive hypothesis: Let $n \geq 1$ and assume $8^n - 2^n$ is a multiple of 6, i.e., there exists an integer $k$ such that $8^n - 2^n = 6k$. Inductive step: We want to show that $8^{n+1} - 2^{n+1}$ is a multiple of 6 as well.

\[ 8^{n+1} - 2^{n+1} = 8(8^n - 2^n) + 8 \cdot 2^n - 2 \cdot 2^n \]
\[ = 8(6k) + 6(2^n) \quad \text{(by the inductive hypothesis)} \]
\[ = 6(8k + 2^n), \]

which is a multiple of 6. Hence, by induction, $8^n - 2^n$ is a multiple of 6 for all integers $n \geq 1$. 