1. **Lemma:** Let \( f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) by \( f(x_1, x_2) = (2x_1 + x_2, 3x_1 - x_2, 2x_1 + x_2) \) for all reals \( x_1, x_2 \). Then \( f \) is one-to-one.

**Proof:** Let \((x_1, x_2)\) and \((y_1, y_2)\) be elements of \( \mathbb{R} \times \mathbb{R} \) such that \( f(x_1, x_2) = f(y_1, y_2) \). Then \( (2x_1 + x_2, 3x_1 - x_2, 2x_1 + x_2) = (2y_1 + y_2, 3y_1 - y_2, 2y_1 + y_2) \), and hence it follows that \( 2x_1 + x_2 = 2y_1 + y_2 \) and \( 3x_1 - x_2 = 3y_1 - y_2 \). Adding these two equations, we find that \( 5x_1 = 5y_1 \) and therefore \( x_1 = y_1 \). Similarly, subtracting 3 times the first equation from twice the second equation, we find

\[
3(2x_1 + x_2) - 2(3x_1 - x_2) = 3(2y_1 + y_2) - 2(3y_1 - y_2).
\]

Simplifying, we find that \( 5x_2 = 5y_2 \) so \( x_2 = y_2 \). Therefore, whenever \( f(x_1, x_2) = f(y_1, y_2) \), it must be that \((x_1, x_2) = (y_1, y_2)\) and so \( f \) is one-to-one.

2. **Theorem:** Let \( A, B, C \) be any sets. Then

\[
[(A \cap B) = C] \Rightarrow [(A \cup C) = A].
\]

**Proof:** We will show that \( A \cup C = A \) by first showing that \( A \subseteq A \cup C \) and then that \( A \cup C \subseteq A \) assuming that \( A \cap B = C \). The first part is immediate since for any element \( x \in A \), we also have that \( x \in A \cup C \) (since \( x \in (A \cup C) \) means \( x \in A \) or \( x \in C \) and we know \( x \in A \)).

For the other direction, we assume that \( x \in (A \cup C) \) so \( x \in A \) or \( x \in C \). If \( x \in C \), then \( x \in A \cap B \) (because \( C = A \cap B \)), which implies that \( x \in A \) and \( x \in B \). Thus, both cases \( (x \in A \) or \( x \in C) \) imply that \( x \in A \), so \( A \cup C \subseteq A \).

Together these two directions demonstrate that \( A = A \cup C \).

3. **Lemma:** The sum of 3 consecutive integers is divisible by 3.

**Proof:** Let \( x, x + 1, x + 2 \in \mathbb{Z} \) be any three consecutive integers. Then their sum can be written as

\[
x + (x + 1) + (x + 2) = 3x + 3 = 3(x + 1).
\]

Since \( x + 1 \) is an integer, the sum \( 3(x + 1) \) must be divisible by 3.
4.a) Prove this theorem:

**Theorem 1.** \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \ [x^2 = y - 1]. \)

**Proof:** Let \( x \in \mathbb{R}. \) Let \( y = 1 + x^2 \) which is in \( \mathbb{R} \) since the reals are closed under multiplication and addition. Therefore, since \( x \) was an arbitrary element in \( \mathbb{R} \) we have proven the theorem.

b) Give a counterexample which disproves the following conjecture when the quantifiers are switched:

**Conjecture 1.** \( \exists y \in \mathbb{R} \forall x \in \mathbb{R} \ [x^2 = y - 1]. \)

To disprove this conjecture we need to show the negation, namely \( \forall y \in \mathbb{R} \exists x \in \mathbb{R} \) such that \( x^2 \neq y - 1. \) Let \( y \in \mathbb{R}. \) Let \( x = \sqrt{y}. \) Notice that \( x^2 \neq y - 1. \) We have shown that for all \( y \in \mathbb{R} \) we can find an \( x \in \mathbb{R} \) such that the proposition is false, it must be that the conjecture is false.

5. **Theorem:** \( n^4 - n^2 \) is divisible by 3 for all \( n \in \mathbb{N}. \)

**Proofs:** If \( n \in \mathbb{N}, \) then either \( n = 3x \) or \( n = 3x + 1 \) or \( n = 3x + 2, \) for some \( x \in \mathbb{Z}. \)

**Case 1:** If \( n = 3x, \) then \( n^4 - n^2 = 81x^4 - 9x^2 = 3(27x^4 - 3x^2), \) which is divisible by 3.

**Case 2:** If \( n = 3x + 1, \) then
\[
    n^4 = n^2 = 81x^4 + 108x^3 + 54x^2 + 9x + 1 - (9x^2 + 6x + 1)
    = 3(27x^4 + 36x^3 + 15x^2 + x),
\]
which is divisible by 3.

**Case 3:** If \( n = 3x_2, \) then
\[
    n^4 = n^2 = 81x^4 + 216x^3 + 216x^2 + 72x + 16 - (9x^2 + 12x + 4)
    = 3(27x^4 + 72x^3 + 68x^2 + 20x + 4),
\]
which is also divisible by 3.

Since we have shown it in all three cases, it follows that \( n^4 - n^2 \) is always divisible by 3.