CS 1050 Practice Midterm Solutions

1. Lemma: Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by $f(x_1, x_2) = (2x_1 + x_2, 3x_1 - x_2, 2x_1 + x_2)$ for all reals x_1, x_2. Then f is one-to-one.

Proof: Let (x_1, x_2) and (y_1, y_2) be elements of $\mathbb{R} \times \mathbb{R}$ such that $f(x_1, x_2) = f(y_1, y_2)$. Then $(2x_1 + x_2, 3x_1 - x_2, 2x_1 + x_2) = (2y_1 + y_2, 3y_1 - y_2, 2y_1 + y_2)$, and hence it follows that $2x_1 + x_2 = 2y_1 + y_2$ and $3x_1 - x_2 = 3y_1 - y_2$. Adding these two equations, we find that $5x_1 = 5y_1$ and therefore $x_1 = y_1$. Similarly, subtracting 3 times the first equation from twice the second equation, we find

$$3(2x_1 + x_2) - 2(3x_1 - x_2) = 3(2y_1 + y_2) - 2(3y_1 - y_2).$$

Simplifying, we find that $5x_2 = 5y_2$ so $x_2 = y_2$. Therefore, whenever $f(x_1, x_2) = f(y_1, y_2)$, it must be that $(x_1, x_2) = (y_1, y_2)$ and so f is one-to-one.

2. Theorem: Let A, B, C be any sets. Then

$$[(A \cap B) = C] \;\Rightarrow\; [(A \cup C) = A].$$

Proof: We will show that $A \cup C = A$ by first showing that $A \subseteq A \cup C$ and then that $A \cup C \subseteq A$ assuming that $A \cap B = C$. The first part is immediate since for any element $x \in A$, we also have that $x \in A \cup C$ (since $x \in (A \cup C)$ means $x \in A$ or $x \in C$ and we know $x \in A$).

For the other direction, we assume that $x \in (A \cup C)$ so $x \in A$ or $x \in C$. If $x \in C$, then $x \in A \cap B$ (because $C = A \cap B$), which implies that $x \in A$ and $x \in B$. Thus, both cases ($x \in A$ or $x \in C$) imply that $x \in A$, so $A \cup C \subseteq A$.

Together these two directions demonstrate that $A = A \cup C$.

3. **Lemma:** The sum of 3 consecutive integers is divisible by 3.

Proof: Let $x, x+1, x+2 \in \mathbb{Z}$ be any three consecutive integers. Then their sum can be written as

$$x + (x + 1) + (x + 2) = 3x + 3 = 3(x + 1).$$

Since $x + 1$ is an integer, the sum $3(x + 1)$ must be divisible by 3.

4.a) Prove this theorem:

Theorem 1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\ [x^2 = y - 1]$.

Proof: Let $x \in \mathbb{R}$. Let $y = 1+x^2$ which is in $\mathbb{R}$ since the reals are closed under multiplication and addition. Therefore, since x was an arbitrary element in $\mathbb{R}$ we have proven the theorem.

b) Give a counterexample which disproves the following conjecture when the quantifiers are switched:

Conjecture 1. $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}\ [x^2 = y - 1]$.

To disprove this conjecture we need to show the negation, namely $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$ such that $x^2 \neq y-1$. Let $y \in \mathbb{R}$. Let $x = \sqrt{y}$. Notice that $x^2 \neq y - 1$. We have shown that for all $y \in \mathbb{R}$ we can find an $x \in \mathbb{R}$ such that the proposition is false, it must be that the conjecture is false.

5. **Theorem:** $n^4 - n^2$ is divisible by 3 for all $n \in \mathbb{N}$.

Proofs: If $n \in \mathbb{N}$, then either $n = 3x$ or $n = 3x + 1$ or $n = 3x + 2$, for some $x \in \mathbb{Z}$.

<u>Case 1:</u> If $n = 3x$, then $n^4 - n^2 = 81x^4 - 9x^2 = 3(27x^4 - 3x^2)$, which is divisible by 3.

<u>Case 2:</u> If $n = 3x + 1$, then

$$n^4 = n^2 = 81x^4 + 108x^3 + 54x^2 + 9x + 1 - (9x^2 + 6x + 1)$$

$$= 3(27x^4 + 36x^3 + 15x^2 + x),$$

which is divisible by 3.

<u>Case 3:</u> If $n = 3x_2$, then

$$n^4 = n^2 = 81x^4 + 216x^3 + 216x^2 + 72x + 16 - (9x^2 + 12x + 4)$$

$$= 3(27x^4 + 72x^3 + 68x^2 + 20x + 4),$$

which is also divisible by 3.

Since we have shown it in all three cases, it follows that $n^4 - n^2$ is always divisible by 3.