CS 1050 Midterm 2 Solutions

1a. Definition: $a \equiv b\ (mod\ m) \Leftrightarrow \exists k \in \mathbb{Z}\ [a - b = mk]$. In other words, the difference of a and b is a multiple of m.

Negation: $a \not\equiv b\ (mod\ m) \Leftrightarrow \exists k, r \in \mathbb{Z}[(0 < r < m) \wedge (a - b = km + r)]$. In other words, the difference of a and b is not a multiple of m.

b. Definition: $f : \mathbb{R} \to \mathbb{Z}$ is onto $\Leftrightarrow \forall y \in \mathbb{Z} \exists x \in \mathbb{R}$ $s.t.$ $f(x) = y$. In other words every element in $\mathbb{Z}$ has a preimage in $\mathbb{R}$.

Negation: $f : \mathbb{R} \to \mathbb{Z}$ is not onto $\Leftrightarrow \exists y \in \mathbb{Z} \forall x \in \mathbb{R}$ $s.t.$ $f(x) \neq y$. In other words there is an element in $\mathbb{Z}$ which has no preimage in $\mathbb{R}$.

c. Definition: $A \subset B \cup C \Leftrightarrow (|B \cup C| > |A|) \wedge (\forall x \in A\ [(x \in B) \vee (x \in C)])$. In other words, every element in A is either in B or in C and $B \cup C$ has strictly more elements than A.

Negation: $A \not\subset B \cup C \Leftrightarrow (|B \cup C| \leq |A|) \vee (\exists x \in A\ [(x \notin B) \wedge (x \notin C)]$. In other words, either $B \cup C$ has atmost as many elements as A or there is an element in A that is neither in B nor in C.

2a. $s_1 = x_1 = 1$, $s_2 = x_1 + x_2 = 1 + 3 = 4$, $s_3 = x_1 + x_2 + x_3 = 1 + 3 + 5 = 9$.

b. Guess $s_n = n^2$.

c. To prove that $s_n = n^2$ for all $n \geq 1$. We give a proof by induction on n.

Base Case: $s_1 = x_1 = 1 = 1^2$. So the statement is true for $n = 1$.

Induction Hypothesis: Assume the statement to be true for some $k \geq 1$. That is, $s_k = k^2$.

Induction Step: We have $s_{k+1} = \sum_{i=1}^{i=k+1} x_i = s_k + x_{k+1}$. So, $s_{k+1} = k^2 + 2(k+1) - 1$. (from I.H. and since $x_{k+1} = 2(k+1) - 1$.). Therefore, $s_{k+1} = k^2 + 2k + 1 = (k+1)^2$. Hence we are through by induction and $s_n = n^2$ for all $n \geq 1$.

3a. f is not onto. Proof by contradiction: suppose f is onto. Then consider $(1, 0) \in \mathbb{Z} \times \mathbb{Z}$. Since f is onto there exists $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that,

$$f(x, y) = (1, 0)$$

$$\Leftrightarrow x + y = 1 \text{ and } x - y = 0$$

$\Leftrightarrow x = y$ and $x + y = 1$

$\Leftrightarrow x = y = \frac{1}{2}$

which is impossible since $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Therefore $(1, 0)$ has no preimage in $\mathbb{Z} \times \mathbb{Z}$, which is a contradiction. So, f is not onto.

b. f is one-one. Proof by contradiction: suppose f is not one-one. So, there exist some $(x_1, y_1), (x_2, y_2) \in \mathbb{Z} \times \mathbb{Z}$, $(x_1, y_1) \neq (x_2, y_2)$ such that,

$f(x_1, y_1) = f(x_2, y_2)$

$\Leftrightarrow x_1 + y_1 = x_2 + y_2$ and $x_1 - y_1 = x_2 - y_2$

$\Rightarrow 2x_1 = 2x_2$ and $2y_1 = 2y_2$ (adding and subtracting the equations)

$\Rightarrow x_1 = x_2$ and $y_1 = y_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$,

which is a contradiction to the fact that (x_1, y_1) and (x_2, y_2) are distinct. Therefore, f is one-one.

4. We need to prove $a_n = 3^n + (-1)^n$ for all $n \geq 0$. We will prove it by strong induction on n.

Base Case: There are two base cases here. $a_0 = 2 = 3^0 + (-1)^0$ and $a_1 = 2 = 3^1 + (-1)^1$, therefore the statement is true for $n = 0$ and $n = 1$.

Induction Hypothesis: Assume that the statement is true for all m, $0 \leq m \leq k$, for some $k \geq 1$. That is, $a_m = 3^m + (-1)^m$.

Induction Step: We are given, $a_{k+1} = 2a_k + 3a_{k-1}$ from the recurrence relation.

$\Rightarrow a_{k+1} = 2(3^k + (-1)^k) + 3(3^{k-1} + (-1)^{k-1})$ from our induction hypothesis.

$\Rightarrow a_{k+1} = 2 \cdot 3^k + 3^k + 2 \cdot (-1)^k + 3 \cdot (-1)^{k-1}$

$\Rightarrow a_{k+1} = 3 \cdot 3^k - 2 \cdot (-1)^{k+1} + 3 \cdot (-1)^{k+1}$ since $(-1)^{k-1} = (-1)^{k+1}$

$\Rightarrow a_{k+1} = 3^{k+1} + (-1)^{k+1}$

which is what we want. So we are through, by induction. Therefore, $a_n = 3^n + (-1)^n$ for all $n \geq 0$.

5. We need to prove that for every $a \in \mathbb{Z}^+$, $a^3 - a$ is a multiple of 3.

Proof: We see that $a^3 - a = a(a^2 - 1) = (a - 1)a(a + 1)$. Now, we see that

$a-1, a, a+1$ are three consecutive non-negative integers (since $a \geq 1$). One of them must be a multiple of 3, which makes $(a-1)a(a+1)$ a multiple of 3 as well. We can see that using case analysis as follows,

Case 1: $n = 3k,\ k \geq 1$. We have $(a-1)a(a+1) = (3k-1)3k(3k+1)$ which is a multiple of 3.

Case 2: $n = 3k+1,\ k \geq 0$. We have $(a-1)a(a+1) = 3k(3k+1)(3k-1)$ which is a multiple of 3.

Case 3: $n = 3k+2,\ k \geq 0$. We have $(a-1)a(a+1) = (3k+1)(3k+2)(3k+3) = 3(3k+1)(3k+2)(k+1)$ which is a multiple of 3.

We see that in each case $a^3 - a$ is a multiple of 3. So, $a^3 - a$ is a multiple of 3 for all $a \in \mathbb{Z}^+$.