CS 1050 Homework 7 Solutions

1. Let $P(n)$ be “$4 + 10 + 16 + \cdots + (6n - 2) = n(3n + 1)$.” We need to prove $P(n)$ for all positive integers $n$.

Proof by induction on $n$.

Base Case: $P(1)$ is “$4 = 1 \cdot (3+1)$” which is true by inspection.

Induction Hypothesis: Let $k$ be any positive integer. Assume that $P(k)$ is true.

Induction Step: LHS of $P(k+1)$ is, $4 + 10 + \cdots + (6k - 2) + (6(k + 1) - 2)$

$= k(3k + 1) + (6k + 1) - 2$ (from Induction Hypothesis)

$= 3k^2 + 7k + 4$

$= (k + 1)(3k + 4)$

$= (k + 1)(3(k + 1) + 1)$, which is what we require.

So $P(k+1)$ is true, and $P(n)$ is true for all positive integers by mathematical induction.

2. Let $P(n)$ be “$8^n - 2^n$ is divisible by 6”. We need to prove $P(n)$ for all positive integers $n$.

Proof by induction on $n$.

Base Case: $P(1)$ is “$8 - 2$ is divisible by 6” which is clearly true.

Induction Hypothesis: Let $P(k)$ be true for some integer $k \geq 1$. In other words we have that $8^k - 2^k = 6t$ for some integer $t$. That is, $8^k = 2^k + 6t$ for some integer $t$.

Induction Step: For $k + 1$ we have, $8^{k+1} - 2^{k+1}$

$= 8(8^k) - 2(2^k)$

$= 8(2^k + 6t) - 2(2^k)$ for some integer $t$ (from Induction Hypothesis).

$= 2^k(8 - 2) + 8 \cdot 6t$

$= 6(2^k + 8t)$, which is divisible by 6, and that is what we want.

So $P(k+1)$ is true and $P(n)$ is true for all positive integers $n$ by mathematical induction.
3. Let $P(n)$ be “$5^n - 4n - 1$ is divisible by 16”. We need to prove $P(n)$ for all integers $n \geq 1$.

Base Case: $P(1)$ is “$5 - 4 - 1$ is divisible by 16” which is clearly true.

Induction Hypothesis: Let $P(k)$ be true for some $k \geq 1$. We have that $5^k - 4k - 1$ is divisible by 16. That is, $5^k - 4k - 1 = 16t$ for some integer $t$. In other words, $5^k = 16t + 4k + 1$ for some integer $t$.

Proof by induction on $n$.

Induction Step: For $k + 1$ we have 

$$5^{k+1} - 4(k + 1) - 1 = 5(5^k) - 4k - 5$$

$$= 5(16t + 4k + 1) - 4k - 5 \text{ (from Induction Hypothesis)}$$

$$= 16(5t + k), \text{ which is divisible by 16.}$$

So, $P(k + 1)$ is true. Therefore, by mathematical induction $P(n)$ is true for all integers $n \geq 1$.

4.a Proof of Lemma 1. We have that for all reals $x \geq 4$, $x - 1 \geq 3$.

$$\Rightarrow (x - 1)^2 \geq 9$$
$$\Rightarrow (x - 1)^2 - 2 \geq 7$$
$$\Rightarrow x^2 - 2x - 1 \geq 7$$
$$\Rightarrow 2x^2 - (x + 1)^2 \geq 7$$
$$\Rightarrow 2x^2 \geq (x + 1)^2 + 7$$
$$\Rightarrow (x + 1)^2 \leq 2x^2 \text{ which is what we want.}$$

b. Proof of Theorem 2. Let $P(n)$ be “$n^2 \leq 2^n$”. We need to prove $P(n)$ for all integers $n \geq 4$. We will prove it by induction on $n$.

Base Case: For $P(4)$, we have that $4^2 \leq 2^4$, which is clearly true.

Induction Hypothesis: Assume $P(k)$ to be true for some integer $k \geq 4$. That is, $k^2 \leq 2^k$.

Induction Step: For $P(k + 1)$ we have the RHS to be, $2^{k+1}$

$$= 2(2^k)$$
$$\geq 2(k^2), \text{ since } k^2 \leq 2^k \text{ from induction hypothesis.}$$
$$\geq (k + 1)^2 \text{ from Lemma 1, because } k \text{ is a real number and } k \geq 4.$$
So we have that \((k + 1)^2 \leq 2^{k+1}\). So, \(P(k + 1)\) is true, and by mathematical induction \(P(n)\) is true for all integers \(n \geq 4\).

5. Clearly we can make 10 cents by using a dime. However, using only dimes and quarters we can make any number of cents \(5n\) such that \(n \geq 4\). In fact we will prove for all \(n \geq 4\) the following statement \(P(n): \) “We can make \(5n\) cents using only dimes and quarters, such that if \(n\) is even then we have atleast 2 dimes and if \(n\) is odd then we have atleast one quarter”. We will prove this statement for all integers \(n \geq 4\) by mathematical induction on \(n\).

Base Case: There are two base cases here, one for \(P(4)\) and another for \(P(5)\). For \(P(4)\), clearly using 2 dimes we can make 20 cents. Also, for \(P(5)\), we can make 25 cents using one quarter. So, the base cases are true.

Induction Hypothesis: Let the \(P(k)\) be true for some \(k \geq 5\).

Induction Step: Consider \(P(k + 1)\). We have 2 cases here,

Case 1. \(k + 1\) is odd. So, we have that \(k\) is even. From the induction hypothesis we get that we can make \(5k\) cents using dimes and quarters such that there are atleast 2 dimes. Therefore, we can make \(5k + 5\) cents by removing 2 dimes from the \(5k\) cents and adding a quarter. That is, we can make \(5(k + 1)\) cents using only dimes and quarters such that there is atleast one quarter, so this case is true.

Case 2. \(k + 1\) is even. So, we have that \(k\) is odd. From the induction hypothesis we get that we can make \(5k\) cents using dimes and quarters such that there is atleast one quarter. Therefore, we can make \(5k + 5\) cents by removing a quarter and adding 3 dimes. That is, we can make \(5(k + 1)\) cents using only dimes and quarters such that there are atleast 2 dimes, so this case is also true.

Hence \(P(k+1)\) is true, and so by mathematical induction \(P(n)\) is true for all \(n \geq 4\). Therefore, we can make any \(5n\) cents such that \(n \geq 4\) by using only dimes and quarters, apart from the obvious 10 cents using a single dime.

6. We will prove the following statement \(P(n)\) for all integers \(n \geq 1\), where \(P(n)\) is:“Using 5 and 6 cent stamps we can make \(5n + m\) cent postage for all \(0 \leq m \leq n\) such that there are exactly \(n - m\) 5cent stamps and \(m\) 6cent stamps”. We will prove this using mathematical induction on \(n\).

Base Case: For \(P(1)\) we have that we can make 5 cent postage using one
5 cent stamp and a $5 + 1 = 6$ cent postage using one 6 cent stamp, which is clearly true. So, the base case is true.

Induction Hypothesis: Let $P(k)$ be true for some $k \geq 1$.

Induction Step: We need to prove $P(k+1)$. That is, we need to prove that we can make $5(k+1) + m$ cent postage for all $0 \leq m \leq k+1$ such that there are exactly $(k+1) - m$ 5 cent stamps and $m$ 6 cent stamps. We will divide this into 2 cases.

Case 1. $0 \leq m \leq k$. From our induction hypothesis we have that we can make $5k + m$ cent postage where $0 \leq m \leq k$ such that there are exactly $k - m$ 5 cent stamps and $m$ 6 cent stamps. Adding one 5 cent stamp we can make $5(k+1) + m$ postage with exactly $(k+1) - m$ 5 cent stamps and $m$ 6 cent stamps. Therefore this case is true.

Case 2. $m = k+1$. From our induction hypothesis we have that we can make $5k + k$ postage using no 5 cent stamp and exactly $k$ 6 cent stamps. Adding another 6 cent stamp, we get $5(k+1) + k + 1 = 5(k+1) + m$ cent postage using no 5 cent stamp and $k+1 = m$ 6 cent stamps. Therefore this case is true.

Hence $P(k+1)$ is true, and so by mathematical induction $P(n)$ is true for all $n \geq 1$. Looking carefully we see that we can construct any postage worth $N$ cents for $N \geq 20$, apart from postage worth 5, 6, 10, 11, 12, 15, 16, 17 and 18 cents.

7. Let $P(n)$ be \(1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6\). We need to prove $P(n)$ for all integers $n \geq 1$. We will prove this by induction on $n$.

Base Case: LHS of $P(1)$ is 1. RHS of $P(1)$ is $(1 \times 2 \times 3)/6 = 1$, which is equal to LHS. Therefore $P(1)$ is true.

Induction Hypothesis: Let $P(k)$ to be true for some $k \geq 1$.

Induction Step: LHS of $P(k+1)$ is $1^2 + 2^2 + \ldots + k^2 + (k+1)^2$
\[
= n(k + 1)(2k + 1)/6 + (k + 1)^2 \text{ (from induction hypothesis)}
\]
\[
= (k + 1)[2k^2 + k + 6k + 6]/6
\]
\[
= (k + 1)[2k^2 + 4k + 3k + 6]/6
\]
\[
= (k + 1)[2k(k + 2) + 3(k + 2)]/6
\]
\[
= (k + 1)(k + 2)(2k + 3)/6
\]
\[(k + 1)((k + 1) + 1)(2(k + 1) + 1)/6,\] which is the RHS of \(P(k + 1)\).
Hence \(P(k + 1)\) is true. Therefore, by mathematical induction \(P(n)\) is true for all integers \(n \geq 1\).