CS 1050 Homework 7 Solutions

1. Let $P(n)$ be "$4+10+16+\cdots+(6n-2)=n(3n+1)$." We need to prove $P(n)$ for all positive integers n.

Proof by induction on n.

Base Case: $P(1)$ is "4 = 1·(3+1)" which is true by inspection.

Induction Hypothesis: Let k be any positive integer. Assume that $P(k)$ is true.

Induction Step: LHS of $P(k+1)$ is, $4+10+\cdots+(6k-2)+(6(k+1)-2)$

$= k(3k+1)+(6(k+1)-2)$ (from Induction Hypothesis)

$= 3k^2+7k+4$

$= (k+1)(3k+4)$

$= (k+1)(3(k+1)+1)$, which is what we require.

So $P(k+1)$ is true, and $P(n)$ is true for all positive integers by mathematical induction.

2. Let $P(n)$ be "$8^n - 2^n$ is divisible by 6". We need to prove $P(n)$ for all positive integers n.

Proof by induction on n.

Base Case: $P(1)$ is "8 - 2 is divisible by 6" which is clearly true.

Induction Hypothesis: Let $P(k)$ be true for some integer $k \geq 1$. In other words we have that $8^k - 2^k = 6t$ for some integer t. That is, $8^k = 2^k + 6t$ for some integer t.

Induction Step: For $k+1$ we have, $8^{k+1} - 2^{k+1}$

$= 8(8^k) - 2(2^k)$

$= 8(2^k + 6t) - 2(2^k)$ for some integer t (from Induction Hyposthesis).

$= 2^k(8-2) + 8 \cdot 6t$

$= 6(2^k + 8t)$, which is divisible by 6, and that is what we want.

So $P(k+1)$ is true and $P(n)$ is true for all positive integers n by mathematical induction.

3. Let $P(n)$ be "$5^n - 4n - 1$ is divisible by 16". We need to prove $P(n)$ for all integers $n \geq 1$.

Base Case: $P(1)$ is "5 - 4 - 1 is divisible by 16" which is clearly true.

Induction Hypothesis: Let $P(k)$ be true for some $k \geq 1$. We have that $5^k - 4k - 1$ is divisible by 16. That is, $5^k - 4k - 1 = 16t$ for some integer t. In other words, $5^k = 16t + 4k + 1$ for some integer t.

Proof by induction on n.

Induction Step:For $k + 1$ we have $5^{k+1} - 4(k + 1) - 1$

$= 5(5^k) - 4k - 5$

$= 5(16t + 4k + 1) - 4k - 5$ (from Induction Hypothesis)

$= 16(5t + k)$, which is divisible by 16.

So, $P(k + 1)$ is true. Therefore, by mathematical induction $P(n)$ is true for all integers $n \geq 1$.

4.a Proof of Lemma 1. We have that for all reals $x \geq 4$, $x - 1 \geq 3$.

$\Rightarrow (x - 1)^2 \geq 9$

$\Rightarrow (x - 1)^2 - 2 \geq 7$

$\Rightarrow x^2 - 2x - 1 \geq 7$

$\Rightarrow 2x^2 - (x + 1)^2 \geq 7$

$\Rightarrow 2x^2 \geq (x + 1)^2 + 7$

$\Rightarrow (x + 1)^2 \leq 2x^2$ which is what we want.

b. Proof of Theorem 2. Let $P(n)$ be "$n^2 \leq 2^n$". We need to prove $P(n)$ for all integers $n \geq 4$. We will prove it by induction on n.

Base Case: For $P(4)$, we have that $4^2 \leq 2^4$, which is clearly true.

Induction Hypothesis: Assume $P(k)$ to be true for some integer $k \geq 4$. That is, $k^2 \leq 2^k$.

Induction Step: For $P(k + 1)$ we have the RHS to be, 2^{k+1}

$= 2(2^k)$

$\geq 2(k^2)$, since $k^2 \leq 2^k$ from induction hypothesis.

$\geq (k + 1)^2$ from Lemma 1, because k is a real number and $k \geq 4$.

So we have that $(k+1)^2 \leq 2^{k+1}$. So, $P(k+1)$ is true, and by mathematical induction $P(n)$ is true for all integers $n \geq 4$.

5. Clearly we can make 10 cents by using a dime. However, using only dimes and quarters we can make any number of cents $5n$ such that $n \geq 4$. In fact we will prove for all $n \geq 4$ the following statement $P(n)$: "We can make $5n$ cents using only dimes and quarters, such that if n is even then we have atleast 2 dimes and if n is odd then we have atleast one quarter". We will prove this statement for all integers $n \geq 4$ by mathematical induction on n.

Base Case: There are two base cases here, one for $P(4)$ and another for $P(5)$. For $P(4)$, clearly using 2 dimes we can make 20 cents. Also, for $P(5)$, we can make 25 cents using one quarter. So, the base cases are true.

Induction Hypothesis: Let the $P(k)$ be true for some $k \geq 5$.

Induction Step: Consider $P(k+1)$. We have 2 cases here,

Case 1. $k+1$ is odd. So, we have that k is even. From the induction hypothesis we get that we can make $5k$ cents using dimes and quarters such that there are atleast 2 dimes. Therefore, we can make $5k+5$ cents by removing 2 dimes from the $5k$ cents and adding a quarter. That is, we can make $5(k+1)$ cents using only dimes and quarters such that there is atleast one quarter, so this case is true.

Case 2. $k+1$ is even. So, we have that k is odd. From the induction hypothesis we get that we can make $5k$ cents using dimes and quarters such that there is atleast one quarter. Therefore, we can make $5k+5$ cents by removing a quarter and adding 3 dimes. That is, we can make $5(k+1)$ cents using only dimes and quarters such that there are atleast 2 dimes, so this case is also true.

Hence $P(k+1)$ is true, and so by mathematical induction $P(n)$ is true for all $n \geq 4$. Therefore, we can make any $5n$ cents such that $n \geq 4$ by using only dimes and quarters, apart from the obvious 10 cents using a single dime.

6. We will prove the following statement $P(n)$ for all integers $n \geq 1$, where $P(n)$ is:"Using 5 and 6 cent stamps we can make $5n+m$ cent postage for all $0 \leq m \leq n$ such that there are exactly $n-m$ 5cent stamps and m 6cent stamps". We will prove this using mathematical induction on n.

Base Case: For $P(1)$ we have that we can make 5 cent postage using one

5cent stamp and a $5+1=6$ cent postage using one 6cent stamp, which is clearly true. So, the base case is true.

Induction Hypothesis: Let $P(k)$ be true for some $k \geq 1$.

Induction Step: We need to prove $P(k+1)$. That is, we need to prove that we can make $5(k+1)+m$ cent postage for all $0 \leq m \leq k+1$ such that there are exactly $(k+1)-m$ 5cent stamps and m 6cent stamps. We will divide this into 2 cases.

Case 1. $0 \leq m \leq k$. From our induction hypothesis we have that we can make $5k+m$ cent postage where $0 \leq m \leq k$ such that there are exactly $k-m$ 5cent stamps and m 6cent stamps. Adding one 5cent stamp we can make $5(k+1)+m$ postage with exactly $(k+1)-m$ 5cent stamps and m 6cent stamps. Therefore this case is true.

Case 2. $m = k+1$. From our induction hypothesis we have that we can make $5k+k$ postage using no 5cent stamp and exactly k 6cent stamps. Adding another 6cent stamp, we get $5(k+1)+k+1 = 5(k+1)+m$ cent postage using no 5cent stamp and $k+1=m$ 6cent stamps. Therefore this case is true.

Hence $P(k+1)$ is true, and so by mathematical induction $P(n)$ is true for all $n \geq 1$. Looking carefully we see that we can construct any postage worth N cents for $N \geq 20$, apart from postage worth 5,6,10,11,12,15,16,17 and 18 cents.

7. Let $P(n)$ be "$1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6$". We need to prove $P(n)$ for all integers $n \geq 1$. We will prove this by induction on n.

Base Case: LHS of $P(1)$ is 1. RHS of $P(1)$ is $(1 \times 2 \times 3)/6 = 1$, which is equal to LHS. Therefore $P(1)$ is true.

Induction Hypothesis: Let $P(k)$ to be true for some $k \geq 1$.

Induction Step: LHS of $P(k+1)$ is $1^2+2^2+\ldots+k^2+(k+1)^2$

$$= n(k+1)(2k+1)/6 + (k+1)^2 \text{ (from induction hypothesis)}$$

$$= (k+1)[2k^2+k+6k+6]/6$$

$$= (k+1)[2k^2+4k+3k+6]/6$$

$$= (k+1)[2k(k+2)+3(k+2)]/6$$

$$= (k+1)(k+2)(2k+3)/6$$

$= (k+1)((k+1)+1)(2(k+1)+1)/6$, which is the RHS of $P(k+1)$.

Hence $P(k+1)$ is true. Therefore, by mathematical induction $P(n)$ is true for all integers $n \geq 1$.