CS 1050 Homework 10 Solutions

1. We need to prove that f is not $O(g)$. Suppose that f is $O(g)$. Then there are constants c, n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. Let m be any odd integer greater than c and n_0, i.e. $m = \max\{c, n_0\}$ and m is odd. Clearly m exists since c, n_0 are constants. Since m is odd, $g(m) = 1$. Therefore we get,

$$f(m) = m$$
$$\Rightarrow f(m) > c$$
$$\Rightarrow f(m) > c \cdot g(m)$$

which is a contradiction since $m > n_0$. Therefore out initial assumption is wrong. So f is not $O(g)$.

2.a In homework 9, problem 5, we showed that given a positive integer A, for all $n \geq 2A^2$, $n! > A^n$. We will use this fact in our proof here. Suppose that f is $O(g)$. Then there exists constants c, n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. That is, $n! \leq c \cdot 3^n$ for all $n \geq n_0$. Now pick an integer m such that $m > \max\{3c, 2 \cdot 3^2,\ n_0\}$. That is, $m - 1 \geq 2 \cdot 3^2$. We know from our proof in homework 9, that $(m-1)! > 3^{m-1}$. So, we get,

$$f(m) = m \cdot (m-1)!$$
$$\Rightarrow f(m) > m \cdot 3^{m-1}$$
$$\Rightarrow f(m) > 3c \cdot 3^{m-1} \text{ (since } m > 3c)$$
$$\Rightarrow f(m) > c \cdot 3^m = c \cdot g(m).$$

which is a contradiction because $m > n_0$. Therefore our initial assumption is wrong. So f is not $O(g)$.

b. Again, in problem 5 of homework 9, we showed that given a positive integer A, for all $n \geq 2A^2$, $n! > A^n$. Substituting, $A = 3$, we get that for all $n \geq 2 \cdot 3^2 = 18$, $n! > 3^n$, i.e. $f(n) > g(n)$. Therefore g is $O(f)$.

3. Proof. We consider the ratio of $g(n)$ and $f(n)$, as follows,

$$\frac{g(n)}{f(n)} = n^{\beta-\alpha}$$

Now since $n \geq 1$, and $\beta - \alpha \geq 0$, we get that $n^{\beta-\alpha} \geq 1$. Therefore

$$\frac{g(n)}{f(n)} \geq 1 \text{ for all } n \geq 1.$$

$\Rightarrow g(n) \geq f(n)$ for all $n \geq 1$.

which implies f is $O(g)$.

4. Taking the ratio of $f(n)$ and $g(n)$, we get,

$$\frac{f(n)}{g(n)} = \frac{4^n}{2^n} = 2^n$$

Taking limits,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} 2^n = \infty$$

So, the limit does not converge. Therefore f is not $O(g)$.

5. Taking the ratio of $f(n)$ and $g(n)$, we get,

$$\frac{f(n)}{g(n)} = \frac{n\log_2 n}{n} = \log_2 n$$

Taking limits,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \log_2 n = \infty$$

So, the limit does not converge. Therefore f is not $O(g)$.

6a. We have $f(n) = \log_2^3(n) = (\log_2 n)^3$, and $g(n) = \log_e n^3 = (3\log_e 2)\log_2 n$. Taking the ratio of $f(n)$ and $g(n)$ we get,

$$\frac{f(n)}{g(n)} = \frac{(\log_2 n)^3}{(3\log_e 2)\log_2 n} = \frac{(\log_2 n)^2}{3\log_e 2}.$$

Taking the limit,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(\log_2 n)^2}{3\log_e 2} = \infty \qquad \text{(since } 3\log_e 2 \text{ is a constant)}$$

So, the limit does not converge to a constant. Hence, f is not $O(g)$.

b. First we show that h is $O(g)$.

$$\frac{h(n)}{g(n)} = \frac{\log_2 n}{(3\log_e 2)\log_2 n} = \frac{1}{3\log_e 2}$$

Taking the limit,

$$\lim_{n\to\infty} \frac{h(n)}{g(n)} = \lim_{n\to\infty} \frac{1}{3\log_e 2} = \frac{1}{3\log_e 2}$$

which is a constant. Therefore we have that h is $O(g)$. We also have that,

$$\lim_{n\to\infty} \frac{g(n)}{h(n)} = \lim_{n\to\infty} \frac{3\log_e 2}{1} = 3\log_e 2$$

which is a constant. So we also have that g is $O(h)$.