CS 7535

Homework 2

Assigned Thursdsay, October 9, 2014

Due Tuesday, October 21, 2014

1. Consider a random walk on the states $\Omega = \{0, 1, \ldots, n\}$. From state i you move to state $i+1 \pmod n$ with probability $1/2$ and to state 0 with probability $1/2$ (so there are no self loops except at 0). In other words, $P_{i,i+1} = P_{i,0} = 1/2$ for all $0 \leq i < n-1$ and $P_{(n-1),0} = 1$.

   Give a coupling argument to bound the mixing time $\tau_x 1/4$ to within a constant factor of the optimal bound. That is, bound the time it takes for the variation distance $||P(x, \cdot), \pi(\cdot)||_{TV} < 1/4$ from any starting state x.

2. Let n, k be positive integers with $k \leq n/2$ and let the state space Ω be all subsets of $\{1, \ldots, n\}$ of size k. We can define a Markov chain on Ω as follows. With probability $1/2$, do nothing. Otherwise, given a set $S \in \Omega$, randomly pick elements $a \in S$ and $b \notin S$ and move to the set $S \cup \{b\} \setminus \{a\}$.

   1. Show that this Markov chain is ergodic with the uniform stationary distribution.

   2. Use a coupling argument to show that the mixing time (number of transitions) is $O(k \log k)$.

3. In problem 5 of homework 1, we showed that the random transposition shuffle mixed in time $O(n^2)$. Now we are going to consider bounding the mixing time of that chain using canonical paths and flows; unfortunately it gives a worse bound, but it will help you understand how to use this technique! Let the state space Ω be the set of permutations on n items, where we connect two states if they differ by transposing two items (and we add self-loops with probability $1/2$ everywhere). Given $x, y \in \Omega$, we define the path $\gamma_{x,y}$ as follows: for each $k = 1, 2, \ldots n$ in turn, we move card x_k from its current position to its final destination in γ.

1. Use the flow encoding technique (and an appropriate injection) to show that the number of paths $\gamma_{x,y}$ that pass through any particular transition of the Markov chain is at most Ω.

2. Now deduce that the mixing time of this Markov chain is $\tau = O(n^3(n \log n + \log \epsilon^{-1}))$. (Note that these methods are both suboptimal, as the true mixing time is $O(n \log n)$.

4. In order to estimate the number of k-colorings of a graph G with $k > 2\Delta$ we saw that we can use our Markov chain for sampling k-colorings on a sequence of graphs G_i. Because our final result is a telescoping procedure involving m terms, we require that each ratio is within a $1 \pm \epsilon/2m$ factor of its true value with probability at least $1 - 1/4m$. Show that if each term in the computation is a 0/1 random variable with expectation $p > 3/4$, then $t = (16m^2)/3\epsilon^2$ samples are sufficient to estimate each term in the product.

   (Use Chebyshevs inequality and explicity calculate the variance of $X = \sum_i^t x_i$ for our indicator variables x_i).

5. We saw in class that we can estimate the number of perfect matchings of any given size if the number of perfect and near perfect matchings are polynomially related. This was done by using a Markov chain whose state space is all matchings and by given a mathing of size k a weight proportional to λ^k.

   Show that without any conditions on the number of perfect or near-perfect matchings we can use that algorithm to efficiently estimate the number of matchings of relatively large size. In particular, show that if k^* is the size of the maximum matching, then we can estimate the number of matchings of size $k = (1 - \epsilon)k^*$, in time $n^{O(1/\epsilon)}$.