Conductance and Canonical Paths

We use a method introduced by Jerrum and Sinclair to show the Markiv chain induced by our transition algorithm is rapidly mixing. Recall the definition of conductance.

$$\Phi_S = \frac{Q(S,\bar{S})}{\pi(S)},$$

where

$$Q(S,\bar{S}) = \sum_{s_1 \in S, s_2 \in \bar{S}} \pi(s_1) p(s_1, s_2)$$

and

$$\pi(S) = \sum_{s \in S} \pi(s).$$

We reformulate the problem using a combinatoric definition.

$$\pi(S) = \frac{|S|}{|\Gamma|}, \quad p(s_1, s_2) = \frac{1}{2\,|E|}, s_1 \neq s_2;$$

$$C(S,\bar{S}) = \text{number of edges crossing the cut.}$$

$$\Phi_S = \frac{C(S,\bar{S})\,/\,2\,|\Gamma|\,|E|}{|S|\,/\,|\Gamma|} \geq \frac{1}{4\,|E|} \frac{C(S,\bar{S})\,|\Gamma|}{|S|\,|S|}.$$

Aside: The **magnification** $= C(S,\bar{S})\,|\Gamma|\,/\,|S|\,|S|$, and the conductance are related by a factor which is the degree of the Markov chain.

Canonical Paths

We want to show that $\Phi_S > \frac{1}{poly}$. To show this we map all points $(I,F) \in S \times \bar{S}$ to points in $C(S,\bar{S}) \times \Gamma$. Then we show that points in $C(S,\bar{S}) \times \Gamma$ have at most a polynomial number of points (I,F) mapped to it. To do this we construct *Canonical Paths* from I to F. The path is constructed by taking the union of I and F and then "fixing" the union. By fixing we mean resolving inconsistences with being the matchings. This fixing process is systematic and eliminates/adds one edge at a time until we have transformed I to F.

The union of two matchings consists of *alternating paths, alternating cycles, double edges (trivial cycles)* and *isolated points*. We fix alternating paths and cycles "in order".

1) There is an order on paths and cycles.
2) Each cycle or path has a fixed first point.
3) Cycles have direction.

The path from one matching to the other takes the following form:

$$I = M_0 \to M_1 \to M_2 \to \ldots \to M_k = F,$$

where each M_i is a matching. At some point along this path we pass through a cut edge, say $(M_i, M_{i+1}) = (T, T')$, where $T \in S$ and $T' \in \bar{S}$. Given the cut edge plus some point in the state space U and some poly information, we can reconstruct I. This shows we have an injective map $|S|\,\bar{S}| \leq C(S,\bar{S})\,|\Gamma|\,|poly|$.

Say we were moving from F to I following the same rules (Canonical path rules). Let U be the same number of steps away from F as T is from I. At any point we are working on one cycle/path (Note, we fix the cycles and paths in the same order as we did when going from I to F). The cycles before the current cycle being fixed match the original mapping I. Those after the current cycle match F. Put another way $I \cup F$ is identical to $T \cup U$ except for some edges in the current cycle. In this cycle, at most 2 of the edges are not completely determined. In fact, since we know T' the difference is at most one edge. This is what requires the extra poly information.

(See figure)

Introduction to Matchings with Weights

We assign weights to matchings based on the size of the matching,

$$W(M) = \lambda^{|M|}$$

. Given the normalization constant $Z = \sum_M W(M)$, we can give the steady state distribution

$$\pi(M) = \frac{W(M)}{Z}.$$

The only restriction on λ is that it is positive. When $\lambda < 1$ we favor smaller matchings. $\lambda = 1$ reduces to the uniform distribution case. Finally, $\lambda > 1$ favors large matchings. The canonical path argument can be extended to weighted matchings.