Transitions between Matchings

Transitions between matchings are governed by the following scheme.

0) Pick some matching M.
1) With probability 1/2 let $M' = M$,
2) Otherwise pick $e \in_U E$ and construct M' as follows.

$$M' = \begin{cases} M - \{e\} & \text{if } e \in M \\ M \cup \{e\} & \text{if } e \text{ can be added to } M \\ M \cup \{e\} - \{e'\} & \text{if } e' \text{ and } e \text{ share exactly one endpoint and } e' \in M \\ M & \text{otherwise.} \end{cases}$$

Issues regarding the Markov Chain - Checks and Balances

1) **Checks**. We verify that the resultant Markov chain is irreducible and aperiodic.
2) **Detailed Balance**. Because the chain is reversible we have the relation

$$\pi_i p_{ij} = \pi_j p_{ji}.$$

And since our transition matrix is doubly-stochastic, the steady state distribution is uniform.

$$p_{ij} = \frac{1}{2|E|} \Rightarrow \pi_i = \frac{1}{\text{number of matchings}}$$

3) **Mixing Rate.** We have measures for the distance between the conditional and the steady state distributions at time t and the amount of time required for this distance to be sufficiently small.

Variation Distance:

$$\Delta_x(t) \equiv \frac{1}{2} \sum_{y \in \Gamma} |p^t(x,y) - \pi(y)| = \max_{S \subseteq \Gamma} |p^t(x,S) - \pi(S)|,$$

where Γ is the set of all matchings and x is the starting position.

Mixing Time:

$$\tau_x(\epsilon) = \min\{t : \Delta_x(t') \leq \epsilon, \forall t' > t\}.$$

4) **Generators.** An algorithm A is a *uniform generator* for S if it outputs $\sigma \in S$ at least $\frac{1}{poly}$ of the time and the variation distance between the conditional distribution on S and the uniform distribution is less than ϵ. The generator is *fully polynomial* (fpaug) if the running time is poly$(n, log\frac{1}{\epsilon})$.

5) **Conductance** Of a cut measures how fast states on opposite sides communicate. Assume $\pi(S) \geq \frac{1}{2}$.

$$\Phi_S = \frac{Q(S,\bar{S})}{\pi(S)},$$

where $Cut(S,\bar{S})$ is the probability of crossing over from S to $\bar{S}$ in one step.

$$Q(S,\bar{S}) = \sum_{s_1 \in S} \pi(s) \sum_{s_2 \in \bar{S}} p(s_1,s_2).$$

For the entire graph

$$\Phi = \min_{S:\pi(S) \leq 1/2} \Phi_S.$$