Today we define some complexity classes.

Definition 1 **P:** *P = The class of polynomial decidable problems (i.e. there exists an algorithm to decide whether there is a solution to a problem instance which runs in polynomial time.)*

Definition 2 **p-relation:** *A relation $R \subseteq \{0,1\}^* X \{0,1\}^*$ is a $p-relation$ if deciding whether $(x, y) \in R$ can be done in polynomoial time in $|x|$ and $|y|$,* **and** *$|y|$ is polynomial in $|x|$ if $(x, y) \in R$.*

(Note that $\{0,1\}^*$ is a string of zeros and ones).

Examples:
SAT

Instance: A boolean formula Φ in CNF (conjunctive normal form);
Question: Is Φ satisfyable?

Eg, $\Phi = (x_1' \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_4' \wedge x_5') \vee \ldots$

$$\begin{aligned} x &= formula \\ y &= truth\ assignment \end{aligned}$$

(You could verify in polynomial time whether y was a satisfying assignment for $x = \Phi$.)

MATCH

Instance: An undirected graph $G = (V, E)$
Question: Does G contain a perfect matching?

$$\begin{aligned} x &= \text{graph } G \text{ (encoded in binary)} \\ y &= \text{a subset of edges} \end{aligned}$$

(You could verify in polynomial time that y is a perfect matching in the graph G.)

Some Formal Definitions:

Let $L \subseteq \{0,1\}^*$ be a set of binary strings representing a language.

Definition 3 NP: *$L \in NP$ if there exists a p-relation such that there exists y such that $(x, y) \in R$ if and only if $x \in L$, where L is a language.*

We think of y as a "witness" (or a "short proof") of x's membership in L.

Definition 4 $\#P$: *$L \in \#P$ if there exists a p-relation such that $f(x) = |\{y : (x, y) \in R\}|\ \forall x$ (where $f : \{0,1\}^* \to \mathcal{N}$).*

There are problems which are *Complete* for each of these classes, or as hard as any other problem in the class. More formally:

Definition 5 Reducibility: *We say that L_1 is polynomially reducible to L_2 (written $L_1 \leq_p L_2$) if we can solve L_1 in polynomial time given a subroutine for solving L_2.*

Definition 6 Polynomial-Time Equivalence: *$L_1 = L_2$ if $L_1 \leq_p L_2$ and $L_2 \leq_p L_1$*

Definition 7 NP-Complete: *L^* is NP-Complete if $\forall\ L \in NP,\ L \leq_p L^*$.*

SAT is NP-Complete.

Definition 8 # P-Complete: *L^* is # P-Complete if $\forall\ L \in \#P,\ L \leq_p L^*$.*

Both # SAT (the counting version of SAT) and # MATCH (the counting version of MATCH) are $\#P$-Complete. Notice that this is surprising since MATCH can be solved in polynomial time!