Given a bipartite graph, $G = (V, E)$, if we can sample perfect matchings it implies we can approximately count. Further, exactly counting perfect matchings is equivalent to calculating the permanent of a $\{0, 1\}$ matrix, which Valiant showed is $\#P$ complete (and thus expected to be intractible).

Today we will show how to generate from the set of all matchings.

Definition 1 Matching: *A set of edges $M \subset E$ is a matching if every vertex is an endpoint of at most one edge in M.*

We will use a Markov chain whose state space is the set of all matchings. (The size of the state space is typically exponentially large in the size of the original graph.)

Procedure:

Pick some initial matching (this can be the empty matching).

1. With probability 1/2 let $M' = M$ (stay where you are). (This slows down the chain by at most a factor of 2 but will make it easier to analyze.)
2. Otherwise Pick $e \in E$ and let

$$M' = \begin{cases} M - \{e\}, \ if\ e \in M, \\ M \bigcup \{e\}, \ if\ e\ can\ be\ added, \\ M \bigcup \{e\} - \{e'\}\ if\ either\ u\ or\ v\ is\ matched\ in\ M\ by\ e'\ (e = (u, v)), \\ M\ otherwise \end{cases}$$

3. Move to M'.

Repeat steps 1 - 3.

This Markov chain is:

- Irreducubility
- Aperiodicty (From 2)

Remark: Notice that we have constucted our chain to have a symmetric transition matrix, which implies that the chain is doubly stochastic and thus has a uniform stationary distribution. It is easy to see that we also have,

$$\pi_i P_{ij} = \pi_j P_{ji} \ \forall\ i, j$$

These are the detailed balance equations which define time-reversibility.

We are left with the question: What is the mixing rate?

Definition 2 Variation Distance: *The variation distance at time t, starting at x, denoted $\Delta_x(t)$ is*

$$\begin{aligned}\Delta_x(t) &= \frac{1}{2}\sum_{y\in\Gamma}|P^t(x,y)-\pi(y)| \\ &= \max_{S\subseteq\Gamma}|P^t(x,S)-\pi(S)|\end{aligned}$$

Note: Γ is the state space and S is a subset of the state space.

Definition 3 Mixing Time: *The mixing time is defined*

$$\tau_x(\epsilon) = min\{t:\Delta_x(t')\leq\epsilon,\ \forall t'>t\}$$

Computer Science Definitions

Definition 4 Uniform Generator *An algorithm is a uniform generator for S if it outputs $\sigma\in S$ at least $\frac{1}{q(n)}$, for some polynomial q and the variation distance between the conditional distribution on S and the uniform distribution is less than ϵ. (I.e., we get samples from S "often" enough and when we are in S the distribution is uniform.)*

Definition 5 Fully Polynomial Almost Uniform Generator *The generator i is fully polynomial (f.p.a.u.g = fully polynomial almost uniform generator) if the running time is polynomial in both n and* $\log(1/\epsilon)$.

The first major tool we will be using to bound the mixing rate of a Markov chain is the *conductance.*

Definition 6 Conductance of a cut: *Let S be a subset of the state space so that $\pi(S)\leq 1/2$ and let S' be the complement Γ
S. The conductance of the cut $\{S,S'\}$ is*

$$\Phi_S = \frac{Q(S,S')}{\pi(S)},$$

where

$$Q(S,S')=\sum_{s_1\in S}\pi(s_1)\sum_{s_2\in S'}P_{s_1s_2},$$

and

$$\pi(S)=\sum_{s\ inS}\pi(s)\leq 1/2.$$

Definition 7 Conductance of a Markov chain: *The conductance of a Markov chain is*

$$\Phi = \min_{S:\pi(S)\leq 1/2}\Phi_S,$$

where the min is taken over all subsets S of the state space.

Later we will prove the following theorem which tells us that the conductance is a good measure of the mixing rate of a Markov chain:

Theorem 1 $\Delta_x(t) \leq \frac{(1-\Phi)^t}{\pi(x)}$

From this it follows that:

Proposition 1 $\tau_x(\epsilon) \leq 2\Phi^{-2}(log\pi(x) - log\epsilon)$

Note: we want $\Phi \geq \frac{1}{poly}$ so that $\tau_x(\epsilon)$ is polynomial.

For our matching Markov chain (defined above), the uniform distribution will be uniform. This allows us to restate the definition of conductance in terms of the size of sets, rather than the corresponding probability measures.
Let $cut(S, S')$ now is the number of edges crossing the cut. Then $Q(S, S') = cut(S, S')/|\Gamma|$ and $\pi(S) = |S|/|\Gamma|$ (where $|X|$ represents the cardinality of the set X). In addition, since $\pi(S') \geq 1/2$, we have that $2|S'|/|\Gamma| \geq 1$. Putting this together, we find that

$$\Phi_S \geq \frac{cut(S, S')\ |\Gamma|}{2\ |S|\ |S'|}.$$

Since we want the conductance to be bounded below by $1/q(n)$, for some polynomial q, this means that we want

$$|S||S'| \leq cut(S, S')\ |\Gamma|\ q(n)/2.$$