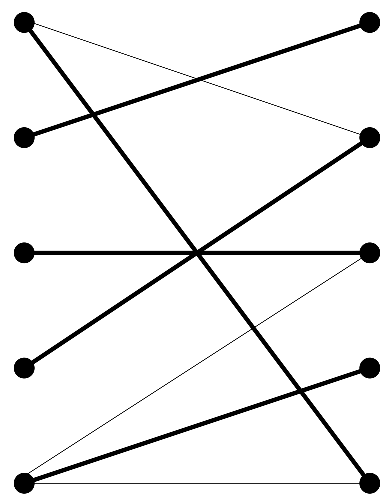

Figure 1: A perfect matching on a bipartite graph

## An Introduction to Sampling and Counting

### Introduction

We begin by discussing perfect matchings on a graph. A graph $G$ is defined by its vertex set and edge set, denoted by $G = (V_1, V_2, E)$. A *bipartite graph* is a graph which has the property $E \subseteq V_1 \times V_2$. For now, we restrict our attention to bipartite graphs with $|V_1| = |V_2| = n$. A *perfect matching* is a set $M \subseteq E$ s.t. $\forall\, u \in V_1 \;\; \exists! \;\; v \in V_2$ s.t. $(u, v) \in M$. In Figure 1, bold lines indicate edges which form a perfect matching.

### Computational Problems

· Input is given as an *adjacency matrix* $A$

$$V_1 = \{1, \ldots, n\}$$
$$V_2 = \{1, \ldots, n\}$$

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

1) Decision Problem
   Input: $G = (V_1, V_2, E)$
   Output: Does $G$ contain a perfect matching?

2) Search Problem
   Input: $G = (V_1, V_2, E)$
   Output: A perfect matching (or it indicates that none exists).

3) Counting
   Input: $G = (V_1, V_2, E)$
   Output: The number of perfect matchings (written in binary).

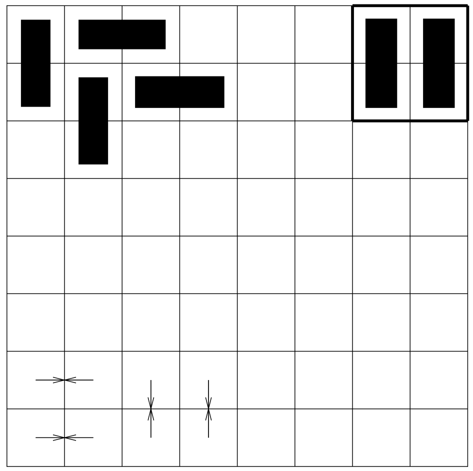

Figure 2: Tiling on a lattice graph

4) Sampling
Input: $G = (V_1, V_2, E)$
Output: A perfect matching chosen uniformly from the set of all perfect matchings (or message saying none exists).

**How many matchings are there?**

Example 1: Complete Graph - denoted by $K(n, n)$.
Number of perfect matchings $= n!$

Example 2: Lattice Graph - $R \subseteq \mathbb{Z}^2$

We can consider the equivalent problem of domino tilings on lattice regions of size $n = 2r \times 2r$ (in the dual graph), see Figure 2. By considering blocks of four cells we obtain the lower bound $2^{r^2} = 2^{n/4}$. We can also consider the "arrow" representation to obtain an upper bound of $4^n$. So

$$2^{n/4} \leq C_n \leq 4^n$$

where $C_n$ is the number of tilings.

**Applications**

1) Statistical Mechanics: the dimer problem
Invented - 1937 (Fowler and Rushbrooke)
Progress - 1961 (Fisher, Kastelyn and Temperley)

a) Approximate Counting
"Estimate" number of solutions
b) Approximate Sampling
Output a matching "close" to the uniform distribution

2) Complexity Theory

$$Det = \sum_{\pi} \left( sgn(\pi) \prod_{i} A_{i\pi(i)} \right)$$

The $Det$ is in $P$, i.e. $\exists$ a polynomial time algorithm to determine it.

$$Perm = \sum_{\pi} \left( \prod_{i} A_{i\pi(i)} \right)$$

The $Perm$ is in $\sharp P$-complete.

**Fact :** the number of perfect matchings $= Perm(A)$.