Introduction

Today, we will extend the results presented last class (for lozenge tilings) to Eulerian orientations (3-colorings with fixed boundary) and to domino tilings.

Quick review from last class

Consider algorithm 2 (with tower moves) for lozenge tilings, described last class. The running time of that algorithm was shown to be $O(n^4)$. In other words, the mixing time of the Markov chain simulated by that algorithm is $O(n^4)$.

This was shown by a coupling argument. Each of the (coupled) Markov chains corresponds to a surface. Let Φ be the volume between these two surfaces. In terms of routings (i.e., non-intersecting lattice paths), Φ is the sum of the areas between corresponding paths.

The proof consisted of proving each of the following statements:
a). Φ is bounded.
b). $E[\Delta\Phi] \leq 0$.
c). If $\Phi_t > 0$ then $E[(\Delta\Phi_t)^2] > v \approx \frac{1}{n}$.
d). If $\Phi_t = 0$ then $\Phi_{t+1} = 0$.

Eulerian orientations (3-colorings with fixed boundary)

Here is an example of an Eulerian orientation and the corresponding routings, given the boundary conditions.

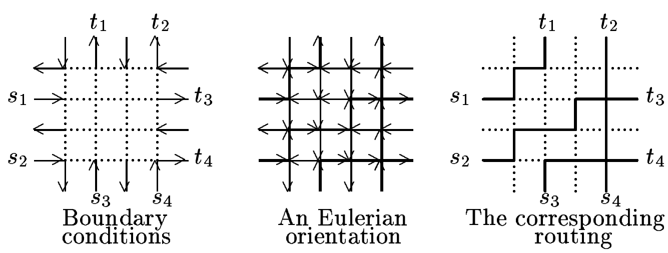
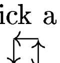
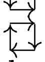
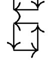


Figure 1: Eulerian orientations and routings

Simple Markov chain:

In terms of Eulerian orientations:

Repeat
Pick a cell.
If  then change to  with probability 1/2.
If  then change to  with probability 1/2.
Otherwise, do nothing.

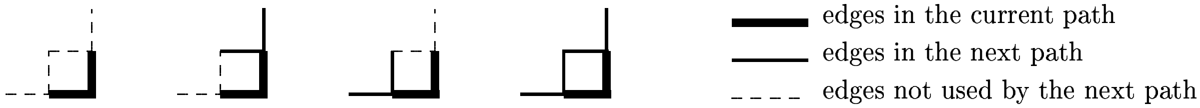
Translating to routing, we would get:

Repeat
Pick a point on the routing.
If ┌ then change (if possible) to ┘ with probability 1/2.
If ┘ then change (if possible) to ┌ with probability 1/2.
Otherwise, do nothing.

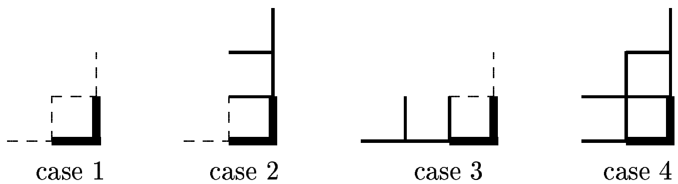
It is open whether or not this is a rapidly mixing Markov chain.

Coming up with another Markov chain for the problem:

There are four possibilities:

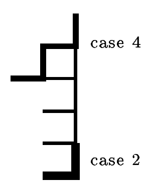


We can interpret these in terms of towers as follows:



If we move in "cascade" (moving ⌟ to ⌜ and also all the "neighbors" to get to a valid configuration), only 1, 2 and 3 will be reversible. After moving in cascade in case 4, there will be no (single) point such that if we move in cascade from that point, we get back to the original configuration. So, if we are in case 4, we will not move.

Observe that these are really the only cases. For example, we cannot have a case 4 on the top of a case 2, as in the picture below. This would be in fact a case 4, because the paths between the one on the top and the bottom one will have to be as in case 4).



Algorithm 2 (with tower moves)

Repeat
  Pick a point on the routing.
  If then change (if possible) to with probability $\frac{1}{2k}$.
  If then change (if possible) to with probability $\frac{1}{2k}$.
  Otherwise, do nothing.

We will see that this Markov chain is rapidly mixing for the same reasons that the lozenge tiling Markov chain is.

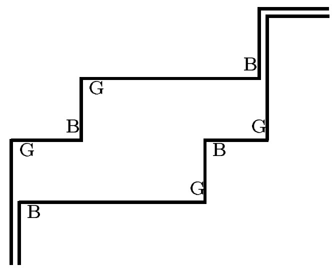
The distance Φ is the sum of the areas between corresponding paths.

As before, we should prove statements a), b), c) and d). But a), b) and d) are similar to a), c) and d) for lozenge tilings. We will prove b): $E[\Delta\Phi] \leq 0$.

One path case:

Label **G** if rotation at that point decreases Φ, and **B** if it increases Φ.

For each "bubble" (see picture below), we have, per path, inside the bubble, a credit of $+$**G**. And at the beginning and end of each bubble, we can have a **B**. Therefore, $\#\mathbf{B} \leq \#\mathbf{G}$ per bubble, and so also in total.



But $E[\Delta\Phi] = (\#\mathbf{B} - \#\mathbf{G})\times$constant. Thus, we have that $E[\Delta\Phi] \leq 0$, completing the proof of statement 2 for the one path case.

Multiple path case:

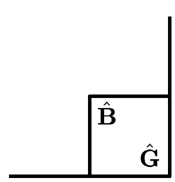
$$\begin{aligned} E[\Delta\Phi] &= \sum_{\text{bad moves}} (\#\text{cells in the tower}) \times (\text{prob. of inverting}) \\ &\quad - \sum_{\text{good moves}} (\#\text{cells in the tower}) \times (\text{prob. of inverting}) \\ &= \sum_{\text{bad moves}} k \times \frac{1}{2kn} - \sum_{\text{good moves}} k \times \frac{1}{2kn} \\ &= \text{constant} \times (\#\mathbf{B} - \#\mathbf{G}), \end{aligned}$$

where n is the number of internal vertices along all paths in any routing.

But now **G** and **B** are defined slightly differently.

Label **G** the moves $\lrcorner$ to $\ulcorner$ which increase Φ and are feasible (cases 1, 2 and 3). Label $\hat{\mathbf{G}}$ the moves $\lrcorner$ to $\ulcorner$ which increase Φ but are not feasible (case 4). Analogously, define **B** and $\hat{\mathbf{B}}$.

Applying the same reasoning as before, we conclude that $\#\mathbf{G} + \#\hat{\mathbf{G}} \geq \#\mathbf{B} + \#\hat{\mathbf{B}}$. But we also have that $\#\hat{\mathbf{G}} \leq \#\hat{\mathbf{B}}$: we can define an injection from $\hat{\mathbf{G}}$'s into $\hat{\mathbf{B}}$'s.



Note that this would not define a bijection, because there might be some extra $\hat{\mathbf{B}}$'s which do not correspond to $\hat{\mathbf{G}}$'s, but which exist because of the boundary restrictions (we cannot move when we are at the boundary).

Domino tilings

There are four types of moves that would cause dominos to cascade up, depending on whether they start and end horizontal or vertical. Similarly, there are four types which cascade down and reverse these moves (see Figure **??**).

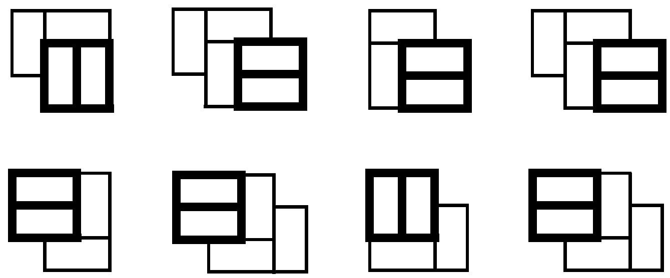


Figure 2: Up are the four types of cascading up moves, and down are their reverse cascading down moves.

Consider the following algorithm:

Repeat
  Pick a 2×2 window.
  If a cascading up move can be done starting at that position, do that with probability proportional to the area covered by the move.
  If a cascading down move can be done starting at that position, do that with probability proportional to the area covered by the move.
  Otherwise, do nothing.

The Markov chain simulated by this algorithm is also rapidly mixing, as the ones described for lozenge tilings and Eulerian orientations.