1. Let $a$, $b$, and $c$ be positive and real numbers. Show that $a^\log_b c = c^{\log_b a}$.

Proof. First note that

$$\log_b c = \log_b c / \log_b a + \log_b a = \log_a c \ast \log_b a.$$ 

Then we have

$$a^{\log_b c} = a^{\log_a c \ast \log_b a} = (a^{\log_a c})^{\log_b a} = c^{\log_b a}.$$ 

2. Let $b$ be a real number greater than 1, and let $x$ and $y$ be positive real numbers. Show that $\log_b (x^y) = y \log_b x$.

Proof. Let $z = \log_b x$. By definition, this means that $b^z = x$. Therefore $b^{zy} = x^y$. Taking the logarithm of both sides, we find that

$$\log_b b^{zy} = \log_b x^y,$$

So

$$zy = \log_b x^y.$$ 

Substituting back for $z$ gives the desired claim.

3. Let $a$ and $b$ be real numbers greater than 1, and let $x$ be a positive real number. Show $\log_a x = \log_b x / \log_b a$. 

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Proof. Let \( \log_a x = u \), so \( x = a^u \). Also, let \( \log_b x = v \), so we have \( x = b^v \) and let \( \log_b a = w \), so \( a = b^w \). It follows that

\[
x = a^u = b^v \Rightarrow (b^w)^u = b^v \Rightarrow b^{wu} = b^v.
\]

But exponentiation is one-to-one, so it follows that \( wu = v \) and therefore \( \log_a x = \log_b x / \log_b a \).

4. Let \( m \) be a positive integer. Show that \( a \equiv b \pmod{m} \) if \( a \equiv b \pmod{m} \).

Proof. If \( a \equiv b \pmod{m} \), then \( a \) and \( b \) have the same remainder when divided by \( m \). Hence \( a = q_1 m + r \) and \( b = q_2 m + r \), where \( 0 \leq r < m \). It follows that \( a - b = (q_1 - q_2)m \) so that \( m | (a - b) \). It follows that \( a \equiv b \pmod{m} \).

5. Let \( m \) be a positive integer. Show that if \( a \equiv b \pmod{m} \) and \( c \equiv d \pmod{m} \) then \( a + c \equiv b + d \pmod{m} \) and \( ac \equiv bd \pmod{m} \).

Proof. Since \( a \equiv b \pmod{m} \) and \( c \equiv d \pmod{m} \), there are integers \( s \), and \( t \) with \( b = a + sm \) and \( d = c + tm \). Hence,

\[
b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)
\]

and

\[
bd = (a + sm)(c + tm) = ac + m(at + cs + stm).
\]

Hence,

\[
a + c \equiv b + d \pmod{m}
\]

and

\[
ac \equiv bd \pmod{m}
\]

6. Find \( 2^{1744} \pmod{127} \).

Notice that \( 2^7 = 128 = 1 \pmod{127} \). Then,

\[
2^{1744} \pmod{127} \equiv 2^{7 \cdot 249} \cdot 2 \pmod{127} \\
\equiv (2^7 \pmod{127})^{249} \cdot 2 \pmod{127} \\
\equiv 1^{249} \cdot 2 \pmod{127} \\
\equiv 2 \pmod{127}.
\]
7. Find the unit’s digit of $287^{3503}$.

First notice that $287^{3503} \pmod{10}$ is the unit’s digit, and this is equivalent to $(287 \pmod{10})^{3503} \equiv 7^{3503} \pmod{10}$.

If we look at successive powers of 7 mod 10, we find $7^0 \equiv 1 \pmod{10}$, $7^1 \equiv 7 \pmod{10}$, $7^2 \equiv 9 \pmod{10}$, $7^3 \equiv 7^2 \cdot 7 \equiv 9 \cdot 7 \equiv 3 \pmod{10}$, and then $7^4 \equiv 7^3 \cdot 7 \equiv 3 \cdot 7 \equiv 1 \pmod{10}$. At this point the sequence $\{1, 7, 9, 3\}$ just repeats for successive powers of 7, so $7^{4k} \equiv 1 \pmod{10}$ for every integer $k$. Therefore,

$$287^{3503} \pmod{10} \equiv 7^{3503} \pmod{10}$$
$$\equiv (7^{4 \cdot 875} \cdot 7^3) \pmod{10}$$
$$\equiv 1^{875} \cdot 7^3 \pmod{10}$$
$$\equiv 3 \pmod{10}.$$

8. What is $3^{602} \pmod{7}$? (Hint: Use Fermat’s little theorem.)

Fermat’s little theorem tells us

$$3^6 \equiv 1 \pmod{7}.$$ 

This tells us that

$$3^{602} \equiv 3^{6 \cdot 100 + 2} \pmod{7}$$
$$\equiv (3^6 \pmod{7})^{100} \cdot 3^2 \pmod{7}$$
$$\equiv 1^{100} \cdot 3^2 \pmod{7}$$
$$\equiv 2 \pmod{7}.$$