**CS 3510 - Honors Algorithms**
**Homework 3/4**
**Assigned February 9**
**Due Tuesday, February 21**

1. a) We are given a directed graph $G$ with weighted edges. Then we are given a shortest path tree for $G$ that is alleged to be a shortest path tree. Give a linear time algorithm that verifies this fact.

   b) One proposed algorithm for finding shortest paths in a directed graph with negative-weighted edges is to make all the weights positive by adding a sufficiently large constant to each weight and then running Dijkstras algorithm. Give an example showing that this method can fail.

   c) We are given a directed graph in which the shortest path between any two vertices $u$ and $v$ is guaranteed to have at most $k$ edges. Give an algorithm that finds the shortest path between two vertices $u$ and $v$ in $O(kE)$ time. (Here edges can have negative weights.)

2. Give an algorithm that will find shortest paths in a directed graph that contains exactly one negative weighted edge. The running time should be on the order of Dijkstras alogrithm.

3. We are given a directed graph $G$ in which the weights of the edges are non-negative integers and the weights are all bounded by $c$. Give an algorithm that finds the shortest path between two vertices $u$ and $v$ in time $O(c|V| + |E|)$. (This time assume all edge weights are positive.)

4. a) A developing country desires to build a network of roads so that a route exists between the capital city and a number of villiages. We are given an undirected graph $G$ in which the edge $(u, v)$ represents the cost of building a road from $u$ to $v$. Give an algorithm that finds a satisfactory network of roads that requires the least cost. (The running time should be at most $O((|V| + |E|)log(|V| + |E|))$.)

   b) How would the cheapest network of roads change if the country moves the capital to one of the villiages?

   c) The bottleneck in a network of roads is the edge that is the most expensive to build. The bottleneck of a graph is the smallest of all the bottlenecks that exist for each possible network of roads that could be built. (In other words, we examine each network that can be built and for each one find the edge that is the worst bottleneck among all of them.) Give an algorithm that finds the bottleneck in a graph. The running time should be at most $O((|V| + |E|)log(|V| + |E|).)$

5. In Problem 2 of problem set 2 you were asked how to go about measuring 13 minutes from three hourglasses that last 11, 7, and 5 minutes each so that the minimum number of turnings is mininized. This time we want to measure 13 minutes in as little time as possible. The answer is still the same: turn 11 and 5 at the same time, start measuring when the 5 minute glass runs out, and turn over the 7 minute glass when the 11 runs out. The total time needed to do this is 11+7=18 minutes. a) Create a problem (i.e., a set of hourglasses and a time interval) for which the answers will be different depending on whether we want to minimize the number of turns or minimize the amount of time elapsed before we can measure the desired time interval. (You do not need to show how to find the answers to your problem and you many use as many hourglasses as you like.) b) How would the algorithm given in the solution to Problem 2 in problem set 2 (or our second homework) be changed so that it finds the answer that uses the least amount of time to measure out the desiired time? Make sure that your algorithm terminates and indicate the running time and prove correctness.

6. An undirected graph $G = (V, E)$ is *bipartite* if and only if its vertices can be partitioned into sets $V_1$ and $V_2$ such that every edges has one vertex in $V_1$ and the other in $V_2$. Give a linear time algorithm based on DFS for determining if a given graph is bipartite.

7. a) Consider an edge-weighted undirected graph $G = (V, E)$ and a particular edge $e$. We wish to find a spanning tree $T$ of $G$ such that (1) $T$ includes the edge $e$, and (2) $T$ has minimal weight over all spanning trees containing $e$. Can this problem be solved by first picking $e$ and then running Prim's algorithm to complete the spanning tree? What about Kruskal's algorithm? Justify your answers.

   b) Let $G$ be a connected undirected graph with the property that every edge has a different weight. Prove that there is only one minimum spanning tree for $G$. (Hint: assume that there are two and examine the lightest edge that is in one tree but not the other to arrive at a contradiction.)

8. Let $T$ be an MST of a graph $G$. Given a connected subgraph $H$ of $G$, show that $T \cap H$ is contained in some MST of $H$. (A subgraph $H = (V_H, E_H)$ of a graph $G = (V_G, E_G)$ has $V_H \subseteq V_G$ and $E_H \subseteq E_G$.)